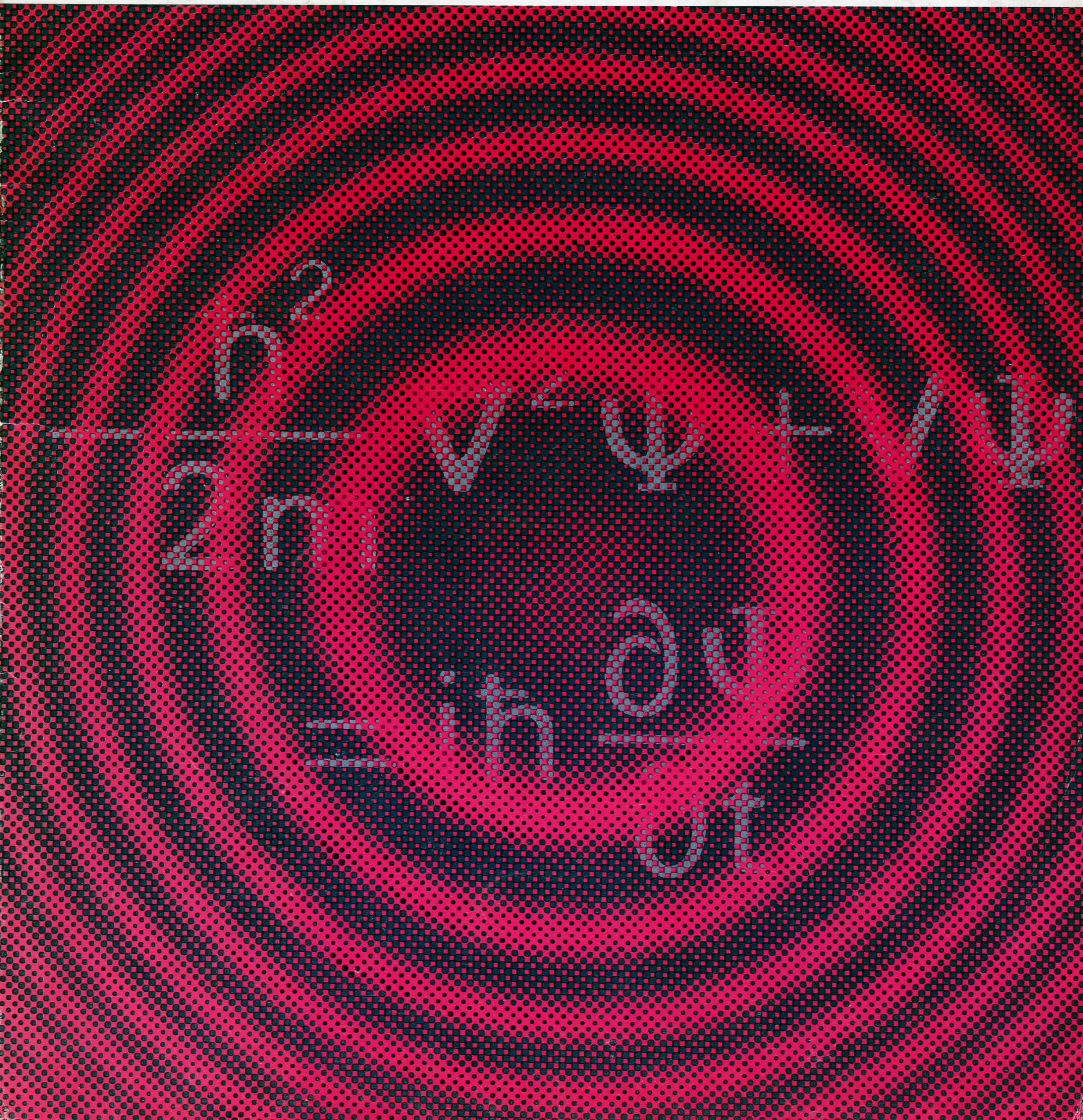
SM351 1

A Third Level Course

THE OPEN UNIVERSITY

Quantum Theory and Atomic Structure

# 1 Classical Mechanics and the Constituents of the Atom



THE OPEN UNIVERSITY

A Third Level Course

# Quantum Theory and Atomic Structure

## Unit 1

## Classical Mechanics and the Constituents of the Atom

*Prepared by a Course Team from the Faculties of Science and Mathematics*

THE OPEN UNIVERSITY PRESS

# SM351

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1.2

# Unit 1

## Classical Mechanics and the Constituents of the Atom

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## Study guide for Unit 1

> Listen to the tape: Unit 1, band 1.

***Before tackling this Unit you should be sure to have read the Study Guide to the Course which is contained in a separate booklet.***

There are two parts to this Unit. The first revises various aspects of classical mechanics. You may find this a little surprising—after all this course is about quantum mechanics not classical mechanics. But quantum mechanics cannot be taught in isolation from the older classical theory. Newton's laws are still perfectly adequate for the solution of a wide variety of day-to-day problems and, at least in these areas, quantum and classical mechanics must have a close correspondence. Accordingly we shall treat classical mechanics not only in the Newtonian manner but also according to a formulation by Hamilton in which the correspondence with quantum mechanics is more direct.

We have severely restricted this résumé so as to cover merely those basic essentials of classical mechanics we shall be requiring in the second part of this Unit and in later Units. In consequence, you may find the treatment a little disjointed. (The alternative would have been to make it more comprehensive—and thereby unnecessarily increase your work load!) You should find this revision comparatively easy going, particularly if you have already studied MST 282.*

Having completed these preliminaries, we are able in the second part of this Unit to begin our study of atomic structure. We begin by taking a look at the constituents of atoms—electrons and nuclei. A substantial treatment is given of the scattering of alpha particles by nuclei (Rutherford scattering); this is the crucial evidence concerning the size of the nucleus. Once again, students of MST 282 will have a comparatively easy time, for they have already made a study of this subject (MST 282, Unit 16).

In the next Unit, we shall continue this first attempt to fashion a model of the atom.

Because the mathematical prerequisites for this Course are *either* MST 282 *or* M 201,** it will take a few Units to bring all students up to the same level of attainment. How you should pace your work over the first four Units will depend on which of these two courses you have previously taken.

*Note to former students of MST 282.*

As you will gather, this first Unit is a rather light one for you. You should note, however, that unless you have studied the optional Unit, MST 282 Unit 12 *Fourier Analysis and Normal Modes*, you will find Unit 4 of the present Course constitutes more than one Unit of work. You are therefore strongly advised to regard this first Unit as being equivalent to approximately three-quarters of a normal Unit; you must be sure to make an early start on Unit 2 so as to allow yourself extra time later for studying Unit 4.

*Note to former students of M 201*

For you, the work-load of this Unit is rather more than a normal one. Compensation for this will come in Unit 4 where you will find quite a lot of material that should already be familiar to you.

* The Open University (1972) MST 282 *Mechanics and Applied Calculus*, The Open University Press.

** The Open University (1972) M 201 *Linear Mathematics*, The Open University Press.

In courses like M 100,* MST 281** and M 201, we were very careful about notation. In this Course, as in MST 282, we shall gain physical clarity by adulterating the mathematical notation. You should therefore begin your studies by now reading Appendix 1 Notation and terminology, pp. 33–7, which is reproduced from MST 282, Unit 1. You will then be in a position to tackle the main body of the Unit.

* The Open University (1971) M 100 *Mathematics: A Foundation Course*, The Open University Press.

** The Open University (1972) MST 281 *Elementary Mathematics for Science and Technology*, The Open University Press.

## General aims and objectives

The general aims of this Unit are to review those aspects of classical mechanics that will be needed later in the Course, and to examine the experimental evidence indicating that atoms are composed of electrons and nuclei.

When you have completed the work for this Unit you should be able to:

1 Define or describe what is meant by the terms: scalar and vector fields; potential, kinetic and total energies; Hamiltonian function; electric and magnetic dipoles; magnetic dipole moment; momentum, angular momentum, torque; electron, nucleus, alpha particle.

2 Define the potential energy function $V(x, y, z)$ as being any scalar function the gradient of which is the negative of a force field $\boldsymbol{F}(x, y, z)$:

$$\boldsymbol{F}(x, y, z) = -\left(\frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial z}\boldsymbol{k}\right)$$

state that if such a scalar exists, the force field is conservative.

3 Find the force field, given a potential energy field, and vice versa; find the force and/or potential energy fields in given simple physical situations.

4 Use Newton's second law in simple cases to find the state of a particle $\{x(t), p(t)\}$ at time $t$, given the initial state $\{x(0), p(0)\}$ and either the one-dimensional force field or one-dimensional potential energy function: $\boldsymbol{F}(x)$ or $V(x)$.

5 Show the equivalence of Hamilton's and Newton's equations of motion where the forces concerned are conservative.

6 Recall the equation relating external torque to rate of change of angular momentum for a system of particles; recognize, for any given set of forces, the existence of a torque about any selected point.

7 Describe in 300–400 words, J. J. Thomson's experiment for measuring the electric charge to mass ratio $(e/m)$ for an electron, deriving or stating the appropriate formulae used in the calculation.

8 Describe in about 300 words the Millikan experiment for measuring the electronic charge, deriving the appropriate formula used in the calculation, *given* Stokes' law.

9 Describe in 200–300 words the experimental apparatus and procedure used in the Rutherford experiment on the scattering of alpha particles by nuclei.

10 Demonstrate your ability to comprehend the various steps in the derivation of the Rutherford scattering formula for the fraction of alpha particles scattered through an angle greater than some value $\phi$. For this purpose you are *not* required to reproduce the whole derivation; instead you should be able to use general principles (such as the conservation of angular momentum, the use of initial conditions, etc.) to develop intermediate steps in the derivation; you should be able to comment upon and relate these parts of the calculation to the overall result.

## Study Sequence for Unit 1

| | Main Text | Set Book |
| --- | --- | --- |
| | | E=EISBERG<br>G=GILLESPIE |
| Review of Classical Mechanics | 1.1.1. Fields | |
| | SAQs 1-2 | |
| | 1.1.2. Newtonian and Hamiltonian formations of mechanics | pp. G29-38 |
| | SAQs 3-14 | |
| | 1.1.3. Dipole fields | |
| | 1.1.4. Angular momentum | |
| | SAQs 15-17 | |
| Constituents of the Atom | 1.2.1. Discovery of the electron | pp.E 70-75 |
| | SAQs 18-20 | |
| | 1. Discovery of the nucleus | pp.E 87-92<br>pp.E 100-108 |
| | SAQs 21-26 | |
| | 1.3. Summary of Unit | |

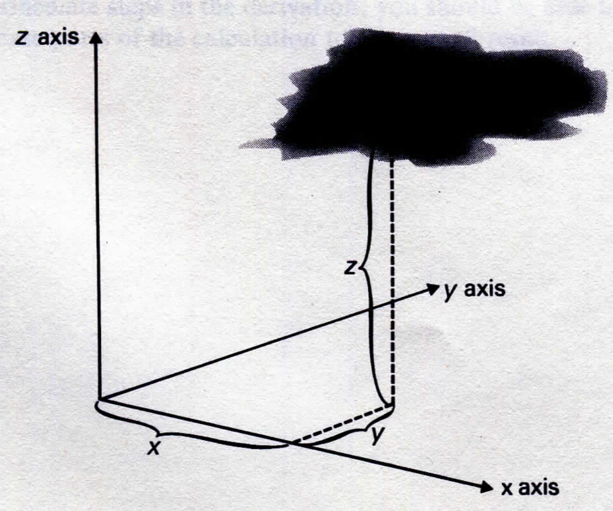



**Figure 1 Definition of subset *D* of space.**

## 1.1 Review of classical mechanics

### 1.1.0 Introduction

Classical mechanics is founded upon Newton's laws of motion. These can be stated in the following familiar terms:

*First law* In an inertial system, every body continues in its state of rest or uniform motion unless acted upon by a force.

*Second law* The net force acting on a particle is proportional to the rate of change of momentum of the particle, i.e. the rate of change of the product: mass times velocity.

*Third law* To every force there is an equal and opposite force, or reaction.

You will note that each law refers to the concept of *force*. In most situations, the force on a particle will depend upon its position in space. It is necessary, therefore, to introduce some kind of function that will summarize the values of the force at each point, and we shall call these functions *fields*. The subject of fields will be our first main topic and it will be dealt with in the next section.

Forces are characterized by both a direction and a magnitude—they are vectors. But vectors are not as easy to handle as scalars, i.e. quantities possessing a magnitude but no direction. We shall find that in many problems it is possible to drop the concept of force and deal instead with a related scalar quantity known as *potential energy*. This can greatly simplify the solution of many problems. We shall find that the introduction of the idea of potential energy leads to the formulation of *the law of conservation of energy* and this, as you probably already know, is a very powerful tool.

Later, we shall develop from the concept of energy a related function called the *Hamiltonian function*. This function assumes great importance later in the Course. Its immediate usefulness is that it allows us to formulate *Hamilton's equations*. These equations represent a formulation of classical mechanics alternative to that originally proposed by Newton.

We should emphasize that the introduction of the concept of energy and of Hamilton's equations adds nothing to what was originally contained in Newton's laws. The same is also true of the introduction of the concept of *angular momentum* with which we shall conclude our brief review of classical mechanics. The importance of these concepts and equations is that they allow us to solve problems economically, and often help our physical insight.

### 1.1.1 Fields

First we recall the meaning of the term *scalar*. This means, roughly, a physical quantity that has a magnitude but no direction associated with it. Examples are temperature, mass and density. Mathematically, we represent a scalar by a real number. (The actual number used to represent a given temperature depends on the system of units chosen, e.g. Fahrenheit or Celsius; so in order to obtain consistent results we must choose a particular system of units and stick to it throughout the calculation.)

By a *scalar field* we mean the set of values of a particular scalar quantity that is defined at every point within a given region of space. For example, we might have a region of space occupied by a block of metal in which the temperature varies from point to point. Mathematically we specify points in three-dimensional space by giving triples of position coordinates, say $(x, y, z)$. The whole of space is represented by the set of all such triples, which is $R^3$ (where $R$ means the set of all real numbers). The part of it we are interested in is represented by some subset of $R^3$, which we shall call $D$ (see Fig. 1).

The scalar field is then represented mathematically by a function, say $\phi$, which associates with each point in $D$ a unique real number in $R$ to represent the value of the scalar at that point. In other words, $\phi$ has domain $D$ and codomain $R$.

The scalar field of most relevance to this Course is the potential energy field (or function). We shall discuss potential energy later in this section.

We shall also need the idea of a *vector field*. The type of vectors we are interested in are those that give the mathematical representation of some directed physical magnitude, such as force*. When this physical vector quantity depends on position, as for example when the force exerted by the Earth's gravitation on a spaceship depends on the position of the spaceship, we call it a vector field. Mathematically, we regard the vectors of a particular type (e.g. forces) as elements of a vector space $V$ (e.g. the set of all possible forces gives a vector space).

We can then regard a vector field as a function which maps the part of real space we are interested in to this vector space; that is, its domain is $D$, and its codomain is the vector space $V$. For example, if $\boldsymbol{F}$ is a vector field then we have, for each $(x, y, z)$ in $D$

$$\boldsymbol{F}(x, y, z) = \text{a vector in } V$$

To represent vectors in $V$ by numbers, we use a basis. We often take as our basis in $V$ the set of three orthogonal vectors in the $x$, $y$ and $z$ directions whose magnitudes, in some chosen system of units, are 1 (e.g. forces of 1 newton in the $x$, $y$, $z$ directions). We denote these basis vectors by $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$.

Then $$\boldsymbol{i}\cdot\boldsymbol{i} = \boldsymbol{j}\cdot\boldsymbol{j} = \boldsymbol{k}\cdot\boldsymbol{k} = 1,$$

and $$\boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\boldsymbol{i} = 0.$$

We can represent an arbitrary vector $\boldsymbol{v}$ in $V$ in the form

$$\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k},$$

where $v_1, v_2, v_3$ are numbers (the coordinates of the vector $\boldsymbol{v}$ with respect to the chosen basis). In particular, if $\boldsymbol{F}$ is a vector field, we can write

$$\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k}$$

where $F_1, F_2, F_3$ are three scalar fields with the same domain as $\boldsymbol{F}$.

An important vector field is the *position vector field*, which associates with each point in $D$ the vector giving the displacement from the chosen origin of coordinates to that point. It is usually denoted by $\boldsymbol{r}$, so that

$$\boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

or, more briefly

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

One use that we can make of the idea of a vector field is to describe the rate at which a given scalar field changes with position. For example, in the theory of heat conduction we want to be able to say how rapidly the temperature in a body is changing with position, because this is one of the factors that determine the flow of heat in the body.

If $\phi$ is any scalar field, then the vector field which gives this spatial rate of change of $\phi$ is called the *gradient* of $\phi$ and is written $\nabla\phi$ ('$\nabla$' being pronounced 'del' or 'grad'); it is defined by

$$\nabla\phi(x, y, z) = \frac{\partial\phi(x, y, z)}{\partial x}\boldsymbol{i} + \frac{\partial\phi(x, y, z)}{\partial y}\boldsymbol{j} + \frac{\partial\phi(x, y, z)}{\partial z}\boldsymbol{k}$$

or, in function notation

$$\nabla\colon \phi \longmapsto \nabla\phi,$$

* The word *vector* is used here in its traditional sense: it describes elements that have both magnitude and direction associated with them. Later in the Course, the same word will be used more generally to describe elements of a more abstract vector space.

where $$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}.$$

We can regard $\nabla$ as a mapping from the set of all (suitably differentiable) scalar fields with given domain $D$ to a set of vector fields with the same domain; this mapping is linear, i.e. it satisfies

$$\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$$

and

$$\nabla(c\phi_1) = c\nabla\phi_1$$

where $c$ is any real (or indeed complex) number. $\nabla$ is therefore a linear operator.

It can be shown that the vector $\nabla\phi(x_0, y_0, z_0)$ at a given point $(x_0, y_0, z_0)$ is directed at right angles to the surface $\phi(x, y, z) = \text{constant}$, passing through $(x_0, y_0, z_0,)$ and has magnitude equal to the rate at which the value of $\phi(x, y, z)$ increases per unit distance at right angles to this surface. For example the scalar field given by

$$\phi(x, y, z) = x^2$$

is constant in the planes $x = \text{constant}$, which are parallel to the $y$-$z$ plane. Its gradient is given by

$$\nabla\phi(x, y, z) = 2x\boldsymbol{i}.$$

This is in the $x$-direction, perpendicular to these planes, and its magnitude $2x$ gives the rate at which the quantity $x^2$ increases with distance as we travel in this direction. If one travelled in some other direction, e.g. the $y$ or $z$ direction, the rate of change of $\phi(x, y, z)$ with distance would be less. *The gradient gives the maximum rate of change with distance.*

You may find it easier to think about this in two dimensions at first. On an ordnance survey map, the height of the land ($\phi$) is a function of the $(x,y)$ position on the map and is therefore a scalar field (see Fig. 2). The gradient of this scalar field at any point on the map is directed at right angles to the contour line through that point, and is thus in the direction of steepest ascent, and its magnitude is equal to the slope of this steepest ascent through that point.

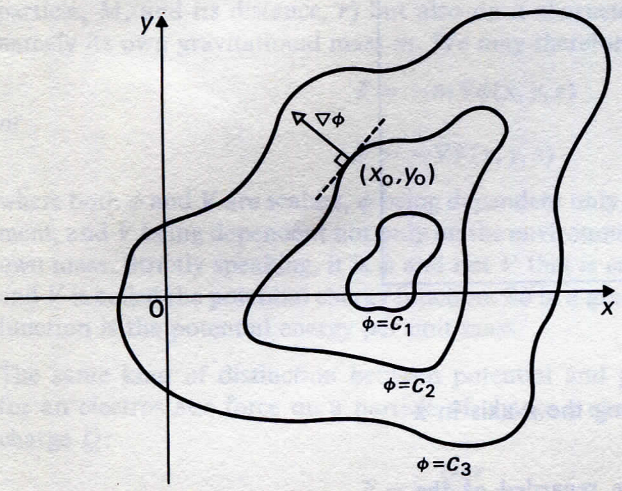



**Figure 2** A contour map of the scalar $\phi$, showing the vector $\nabla\phi$ at the point $(x_0, y_0)$, the vector being perpendicular to the contour through $(x_0, y_0)$.

The vector fields used in this Course will be fields of force. Some forces, such as gravitational or electrostatic forces, can be represented by gradients. The associated scalar field can then be shown to be minus the potential energy field (or potential energy function, as it is often called). In such cases, the force field is minus the gradient of the potential energy, and is said to be a conservative force field. (As you will see later the name 'conservative' arises from the fact that such forces allow the conservation of energy). Some forces, such as frictional forces which depend on the direction of motion, cannot be regarded as force fields and do not have an associated field of potential energy.

Here are two short SAQs to test how well you have followed the material so far:

**SAQ 1** *(related to Objective 3)* **short**

If $\phi$ is a scalar field given by $\phi = 4x + 3y$, find an expression for the gradient of the field.

(Solution on p. 40.)

**SAQ 2** *(related to Objective 3)* **short**

If $\phi$ is a scalar field given by $\phi = 6z^2$, find the gradient at the point $z = 5$.

(Solution on p. 40.)

### 1.1.2 The Newtonian and Hamiltonian formulations of mechanics

Having briefly introduced these ideas on fields, we would now like you to read a passage from the set book *A Quantum Mechanics Primer* by D. T. Gillespie.*

It will revise the concepts of force, potential energy, kinetic energy and momentum. This will be done in the context of Newton's second law. Newton's way of formulating mechanics, as we said in the introduction, is not the only way of doing it—indeed for our purposes it is sometimes not even the most convenient way. An alternative formulation due to Hamilton has its advantages and is closer to the formulation of quantum mechanics, so we take this early opportunity of introducing it. The importance of the Hamiltonian function will become steadily more apparent as the Course progresses.

Now study Gillespie (G) pp. G 29–38 inclusive.

Refer to the additional notes below if you are in difficulties—you may find a helpful comment there. Work through the exercises either when you come to them, or a little later in this text when they are presented to you as SAQs. For your convenience, we list here the pages in this text on which you will find the solutions to the exercises.

| | | Solution |
|---|---|---|
| Exercise 21a | (SAQ 6) | p. 40 |
| Exercise 21b | (SAQ 7) | p. 41 |
| Exercise 21c | (SAQ 8) | p. 41 |
| Exercise 22 | (SAQ 11) | p. 41 |
| Exercise 23 | (SAQ 12) | p. 42 |
| Exercise 24 | (SAQ 13) | p. 42 |

**Additional notes**

*p. G 29, line 9 up*** ... '*m* which is constrained to move along the *x*-axis in a *conservative*' ...

'Conservative' is the name given to a force field that can be regarded as the gradient of a potential energy scalar field.

*p. G 29, line 8 up* ... 'force field, $F(x)$. In order to avoid the *complications of the theory of relativity*' ...

Relativity theory requires that at speeds approaching the speed of light an allowance must be made for a change in mass.

* D. T. Gillespie (1970) *A Quantum Mechanics Primer*, International Textbook Company.

** We shall use this terminology to denote the ninth line from the foot of the page.

*p. G 30, line 1*...'environment. This interaction may also be described by the *potential function* $V(x)$'...

Strictly speaking $V(x)$ is the potential energy function. We shall discuss the difference between potential function and potential energy function at the end of this reading passage.

*p. G 30, line 4* equation (3–2)

This expression refers to the one dimensional case. For the general three-dimensional case, we have

$$\boldsymbol{F} = -\nabla V$$

Note, in both cases, the negative sign.

*p. G 30, line 3 up*...'The basic programme of mechanics, *both classical and quantum*, is'...

This statement is of great importance. There are so many *differences* between classical and quantum mechanics that it is easy to overlook those features they have in common.

*p. G 33, line 1*...'As a familiar example, for the simple force field $F(x) = k$, or'...

An example of such a force is the gravitational force a particle experiences in a uniform gravitational field.

In that passage from Gillespie there was a possible source of confusion over his use of the phrase 'potential function'. It is sometimes important to make a distinction between the *potential function* and the *potential energy function*. For example, the gravitational force exerted by a particle of mass $M$ on another particle of mass $m$ when they are separated by a distance $r$ is given by $\boldsymbol{F} = \dfrac{GMm}{|\boldsymbol{r}|^3}\boldsymbol{r}$ where $G$ is the gravitational constant and $\boldsymbol{r}$ is the position vector of $M$ with respect to $m$. The force on $m$ not only depends upon its environment (i.e. the mass of the neighbouring particle, $M$, and its distance, $r$) but also on a characteristic of the particle itself, namely its own gravitational mass $m$. We may therefore write

$$\boldsymbol{F} = -m\nabla\phi(x, y, z)$$

or

$$\boldsymbol{F} = -\nabla V(x, y, z)$$

where both $\phi$ and $V$ are scalars, $\phi$ being dependent only upon the particle's environment, and $V$ being dependent not only on the environment but also on the particle's own mass. Strictly speaking, it is $\phi$ and not $V$ that is called the potential function, and $V$ is called the potential energy function. So in a gravitational field the potential function is the potential energy per unit mass.

The same kind of distinction between potential and potential energy holds true for an electrostatic force on a particle of charge $q$ exerted by another particle of charge $Q$:

$$\boldsymbol{F} = \frac{Qq\boldsymbol{r}}{4\pi\epsilon_0|\boldsymbol{r}|^3}$$

where $4\pi\epsilon_0$ is a constant. (See the Section on units at the back of the Glossary to the Course.)

This can be written either in terms of the potential function

$$\boldsymbol{F} = -q\nabla\phi(x, y, z)$$

or in terms of the potential energy function

$$\boldsymbol{F} = -\nabla V(x, y, z)$$

So, in this electrostatic case, the potential is the potential energy per unit charge.

On the other hand, in the case of a particle attached to the end of a stretched spring, the force it experiences is independent of the characteristics of the particle itself and depends only upon the characteristics of the spring—its spring constant, $k$, and its extension, $\boldsymbol{x}$:

$$\boldsymbol{F} = -k\boldsymbol{x}$$

Since in this case the force is independent of any property of the particle, the potential function is a less useful construct than the potential energy function.

Another point we wish to make is that the treatment given in Gillespie is confined to the one-dimensional case. Although problems we shall be tackling in this Course will indeed involve motion in only one dimension, others, such as the treatment of the structure of the hydrogen atom, are three-dimensional problems.

Fortunately, the generalization from one dimension to three dimensions is easily made, as follows.

Let us first consider the definition of the work done on a body in moving it from $x_1$ to $x_2$. It is given in equation G (3–4) on p. G 33:

$$W_{12} \equiv \int_{x_1}^{x_2} F(x)\,\mathrm{d}x \qquad \text{G (3–4)}$$

For the three-dimensional case, the integral becomes a *line integral.* This can be explained in the following way:

When no longer restricted to the $x$ axis, the motion at any time might not lie in the same direction as the force—since the force can vary in both magnitude and direction from point to point, and the path of the particle may be curved. Whatever path the particle follows, it will always be possible to divide up that path into lengths that are short enough to be regarded as straight lines. And, so long as the force changes continuously (or remains constant) from one point to the next, one can always find a value for it at a point sitting 'in the middle' of one of these very short lengths (or 'elements') of the particle's path.

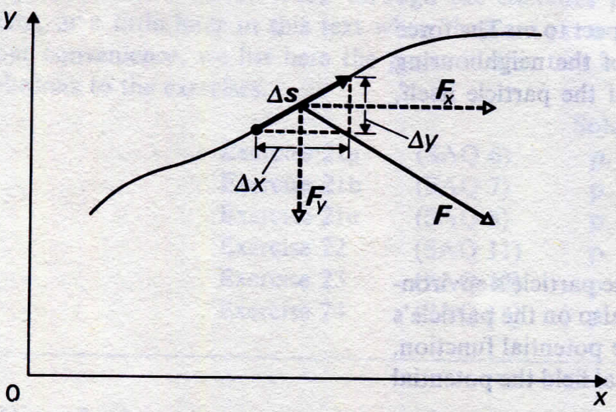



**Figure 3 Calculation of the work done by a force.**

Concentrate for a moment on just one of these elements (see Fig. 3). For the case of *two* dimensions, the element $\Delta \boldsymbol{s}$ can be resolved into two components $\Delta \boldsymbol{x}$ and $\Delta \boldsymbol{y}$. The force acting at the point where this element is located can likewise be resolved into two components $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$.

Then the work done on the particle by the $x$-component of the force, as the particle moves along this element, is the scalar quantity

$$(\Delta W)_x = F_x \Delta x$$

But the particle also moves in the $y$ direction and work is done by the $y$ component of the force:

$$(\Delta W)_y = F_y \Delta y$$

So the total work done on the particle is:

$$\Delta W = (\Delta W)_x + (\Delta W)_y = F_x \Delta x + F_y \Delta y$$

Similarly in *three* dimensions we have

$$\Delta W = F_x \Delta x + F_y \Delta y + F_z \Delta z$$
$$= \boldsymbol{F} \cdot \Delta \boldsymbol{s}$$

where
$$\boldsymbol{F} = F_x \boldsymbol{i} + F_y \boldsymbol{j} + F_z \boldsymbol{k}$$

and
$$\Delta \boldsymbol{s} = \Delta x \boldsymbol{i} + \Delta y \boldsymbol{j} + \Delta z \boldsymbol{k}$$

To find the total work done in taking the particle between the two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, we must add the contribution from all the elements $\Delta \boldsymbol{s}$. This sum is represented in the limit* of $\Delta \boldsymbol{s} \rightsquigarrow \boldsymbol{0}$ by

$$W_{12} = \int_{(x_1,y_1,z_1)}^{(x_2,y_2,z_2)} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

where the integral here is called *a line integral.* This may be written

$$W_{12} = \int_{x_1}^{x_2} F_x \cdot \mathrm{d}x + \int_{y_1}^{y_2} F_y \cdot \mathrm{d}y + \int_{z_1}^{z_2} F_z \cdot \mathrm{d}z$$
$$= \int_{x_1}^{x_2} \left(m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right) \mathrm{d}x + \int_{y_1}^{y_2} \left(m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) \mathrm{d}y + \int_{z_1}^{z_2} \left(m\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right) \mathrm{d}z$$

By analogy with the derivation of equation G (3–5b) p. G 34, this becomes

$$W_{12} = \tfrac{1}{2}mv_x^2\Big|_{x_1}^{x_2} + \tfrac{1}{2}mv_y^2\Big|_{y_1}^{y_2} + \tfrac{1}{2}mv_z^2\Big|_{z_1}^{z_2}$$
$$= \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

where $\boldsymbol{v}_2$ and $\boldsymbol{v}_1$ are now the final and initial velocities in three dimensions. Similarly, when there is a potential energy such that $\boldsymbol{F}(x, y, z) = -\nabla V(x, y, z)$, one can show that:

$$W_{12} = V(x_1, y_1, z_1) - V(x_2, y_2, z_2)$$

(which is similar to equation G (3–5a)).

Thus the total energy of the particle in three dimensions is

$$E = \frac{p^2}{2m} + V(x, y, z)$$

where $\boldsymbol{p}$ is now the momentum in three dimensions, and where, as we have said, we are assuming that we can express $\boldsymbol{F}$ as $-\nabla V$.

It follows that the Hamiltonian function given by equation G (3–8) on p. G 36 now becomes

$$H(x, y, z, p) = \frac{p^2}{2m} + V(x, y, z).$$

(Note that the Hamiltonian function is not quite unique, in the sense that one may always add an arbitrary constant to $V$. This does not usually matter, for in physics one is only interested in *changes* in potential energy.)
This concludes our first set book reading.

You should now be able to meet Objectives 2, 3, 4 and 5 as well as half of Objective 1. Check your understanding of the material so far presented by trying the various Self-assessment questions (SAQs) that relate to the objectives. For your own sake, please do not look up solutions to these or later SAQs until you have made a genuine attempt to answer them yourself.

As you will note from the comment in the right-hand margin opposite each SAQ, these problems (in the view of the Course Team) are of varying degrees of difficulty. The comment—'short', 'medium' and 'long'—will give you some idea of how the Course Team rates the particular SAQ. If you get stuck, look to see whether

* We use $\boldsymbol{0}$ to denote a zero vector.

there is a 'hint on tape' comment in the margin. If there is, use your tape recorder to listen to the hint given on the appropriate tape; then have another go at solving the problem before looking up the answer.

**SAQ 3** (*Objectives 2 and 3*) A potential energy function, $V$, is given by **short**

$$V = 2x + 4y^2 - z^3 + 2$$

What is the force at the point (2, 1, 3)?

(Solution on p. 40.)

**SAQ 4** (*Objectives 2 and 3*) Suppose there are two potential energy functions **short**

$$V_1 = 8x + y^2 + 8z$$
$$V_2 = 2x^2 + 8y + z^2$$

At the point (2, 3, 3), the forces due to the potentials $V_1$ and $V_2$

(i) are identical
(ii) have the same magnitude but different directions
(iii) have the same direction but different magnitudes
(iv) have different magnitudes and directions.

(Solution on p. 40.)

**SAQ 5** (*Objective 3*) A mass is acted upon by two external agencies. One can be described in terms of a potential **short**

$$V_1 = 7x + 4y^2 + z$$

and the other in terms of a potential

$$V_2 = x^3 - y + 6z^2$$

What is the combined potential at the point (3, 4, 5)?

(Solution on p. 40.)

**SAQ 6** (*Objectives 2 and 3*) Exercise 21a p. G 30 **short**

(Solution on p. 40.)

**SAQ 7** (*Objectives 2 and 3*) Exercise 21b p. G 30 **short**

(Solution on p. 41.)

**SAQ 8** (*Objectives 2 and 3*) Exercise 21c p. G 30 **medium**

(Solution on p. 41.)

**SAQ 9** (*Objective 4*) 'In order to specify the time evolution of a classical mechanical system, it is necessary to know the initial state of the system: specification of the state at any other time is not sufficient.' **short**

Is this statement true or false?

(Solution on p. 41.)

**SAQ 10** (*Objective 4*) A particle of unit mass is free to move along the $x$-axis, the force acting on it being **short**

$$F(x) = (6x^2 + 2)\mathbf{i}.$$

If the particle is initially at $x = 0$ at $t = 0$, when will it arrive at $x = 1$?

(i) 0.5 s
(ii) 1 s
(iii) 0.1 s
(iv) 1.5 s
(v) some other value
(vi) there is not enough information to answer the question

(Solution on p. 41.)

**SAQ 11** (*Objective 4*) Exercise 22 p. G 36 **medium**

(Solution on p. 41.)

**SAQ 12** (*Objective 5*) Exercise 23 p. G 37 **medium**

(Solution on p. 42.)

**SAQ 13** (*Objective 5*) Exercise 24 p. G 38 **medium**

(Solution on p. 42.)

**SAQ 14** (*Objective 5*) For a particle of mass $m$ in a uniform gravitational field, the Hamiltonian function is $H(x,p) = \dfrac{p^2}{2m} + mgx$, where $x$ is the height. **long**

At time $t=0$, the particle is released from rest at a height $h$ above the ground, i.e. $[x(0), p(0)]$ is $[h, 0]$. *Starting from Hamilton's equations*, find the time taken for the particle to hit the ground. **hint on tape 1, band 1**

(Solution on p. 43.)

### 1.1.3 The dipole field

While on the subject of fields and forces, let us consider the dipole field. It is of importance when considering the magnetic field associated with an atom. (Actually we shall not need to call upon the material of this Section until Unit 13, where we deal with the magnetic effects of atoms. Nevertheless, we think it best to get this preliminary out of the way now as it is another example of an activity associated with Objectives 1 and 3.)

We begin with an electric dipole. It consists of two equal and opposite electric point charges separated by a distance $d$, say. An *ideal dipole* is one in which the distance $d$ is vanishingly small compared with any distance at which the field is considered, and the point charges $+Q$ and $-Q$ are sufficiently large to preclude the vanishing of the product $Qd$.

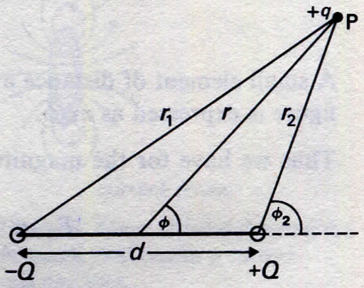



Figure 4 The calculation of the potential energy field due to an electric dipole.

The aim is to find the force field due to the dipole. Our strategy is first to find the scalar potential energy due to each point charge separately, then, assuming linear superposition, to combine them to get the resultant potential energy. Finally, the gradient of this resultant scalar field is found, and this gives the vector force field. To study this dipole, we consider the potential energy of a small test charge, $q$, placed at point P in Figure 4. It will comprise the sum of the potential energies due to the two charges $+Q$ and $-Q$. The contribution due to $+Q$ is $\dfrac{+Qq}{4\pi\epsilon_0 r_2}$. (See the second part of the solution to SAQ 7, (p. 41) where we are now writing the constant $k$ as $Qq/4\pi\epsilon_0$, an expression that will be familiar to you if you have already studied electrostatics as, for example, in S 100*, Unit 4.) The contribution due to $-Q$ is $\dfrac{-Qq}{4\pi\epsilon_0 r_1}$.

Thus we may write for the potential energy

$$V = \frac{Qq}{4\pi\epsilon_0 r_2} - \frac{Qq}{4\pi\epsilon_0 r_1} = \frac{Qq(r_1 - r_2)}{4\pi\epsilon_0 r_1 r_2} \qquad (1)$$

But $r_1{}^2 = r_2{}^2 + d^2 + 2r_2 d\cos\phi_2$ (law of cosines).

So

$$\begin{aligned} r_1{}^2 - r_2{}^2 &= d^2 + 2r_2 d\cos\phi_2 \\ (r_1 - r_2)(r_1 + r_2) &= d(d + 2r_2\cos\phi_2) \\ (r_1 - r_2) &= d(d + 2r_2\cos\phi_2)/(r_1 + r_2) \qquad (2) \end{aligned}$$

Thus far, the calculation is exact and applies to any dipole, i.e. for any $d$ large or small compared with $r$.

***Before reading on, try to find a simplified form of equation 1 for an ideal dipole.***

* The Open University (1971) S 100 *Science: A Foundation Course*, The Open University Press.

For an ideal dipole $d \ll r_1, d \ll r_2$. Thus, in equation 2, the quantity $(d+2r_2 \cos \phi_2)$ can be approximated by $2r_2 \cos \phi_2$. Also $r_1 \approx r_2 \approx r$, and $\phi_2 \approx \phi$. Equation 2 then simplifies to

$$(r_1 - r_2) = 2rd \cos \phi / 2r = d \cos \phi$$

and equation 1 becomes

$$V = \frac{Qqd \cos \phi}{4\pi\epsilon_0 r^2} \qquad (3)$$

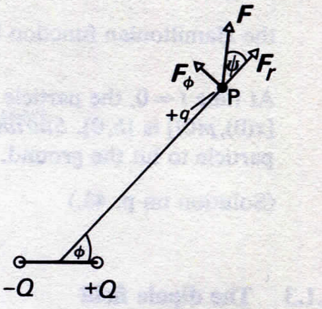



Figure 5 The calculation of the force field due to an electric dipole.

This gives the scalar potential energy field for small $d$. In order for us to arrive at the force vector field, $V$ must be differentiated. The magnitude of the component of the force in any direction $x$ is given by

$$F_x = -\frac{\partial V}{\partial x} \qquad \text{(see equation G (3–2) p. G 30)}$$

In the plane of Figure 4, it is convenient to use polar coordinates and express the force at P in terms of two components: one of them, $F_r$, acting along the radius vector, $\mathbf{r}$, and the other $F_\phi$, at right angles to it and lying in the direction of increasing $\phi$ (see Fig. 5).

The magnitude of $F_r$ is obtained by finding the negative of the rate of change of $V$ along $\mathbf{r}$:

$$F_r = \lim_{h \to 0} \left[ -\frac{V(r+h, \phi) - V(r, \phi)}{h} \right] = -\frac{\partial V(r, \phi)}{\partial r}$$

So, substituting for $V$ from equation 3, we get

$$F_r = \frac{2Qqd \cos \phi}{4\pi\epsilon_0 r^3} \qquad (4a)$$

A small element of distance at right angles to the radius vector in the plane of the figure is expressed as $r\Delta\phi$.

Thus we have for the magnitude of the second component of the force:

$$F_\phi = \lim_{\Delta\phi \to 0} \left[ -\frac{V(r, \phi+\Delta\phi) - V(r, \phi)}{r\Delta\phi} \right]$$

$$F_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} = \frac{Qqd \sin \phi}{4\pi\epsilon_0 r^3} \qquad (4b)$$

*Before reading on, answer the following questions. What is the magnitude of the resultant force? What angle does it make to* $\mathbf{r}$*?*

The magnitude of the resultant force is given by

$$F = \left[ \left( \frac{2Qqd \cos \phi}{4\pi\epsilon_0 r^3} \right)^2 + \left( \frac{Qqd \sin \phi}{4\pi\epsilon_0 r^3} \right)^2 \right]^{1/2}$$

$$= \frac{Qd}{4\pi\epsilon_0 r^3} q \, [4 \cos^2 \phi + \sin^2 \phi]^{1/2}$$

$$= \frac{Qd}{4\pi\epsilon_0 r^3} q \, [1 + 3 \cos^2 \phi]^{1/2} \qquad (5a)$$

Note that the characteristic property of the dipole that determines the strength of the force is the product $Qd$. Although for the ideal dipole the distance $d$ is considered vanishingly small, the product $Qd$ remains finite.

The angle $\psi$ that the resultant force makes to the radius vector $\mathbf{r}$ (see Fig. 5) is given by

$$\tan \psi = F_\phi / F_r$$

$$= \sin \phi / 2 \cos \phi$$

$$= \tfrac{1}{2} \tan \phi \qquad (5b)$$

Equations 5a and b allow one to investigate the ideal dipole force field in a plane. In fact, it also gives us the field in three dimensions because of the axial symmetry about the line between $+Q$ and $-Q$, i.e. there is no dependence on the azimuthal angle about this line.

In order to visualize the field, it is useful to introduce the idea of lines of force. These are continuous lines in space such that the direction of $\boldsymbol{F}$ at any point lies along the tangent to the line. The density of lines (the number crossing unit area perpendicular to the lines) in a region of space is chosen to be proportional to the magnitude of the force in that region.

In Figures 6a and b, we use lines of force to indicate the fields of force due to an ordinary dipole and to an ideal dipole. The arrows give the sense of the force on a positive test charge.

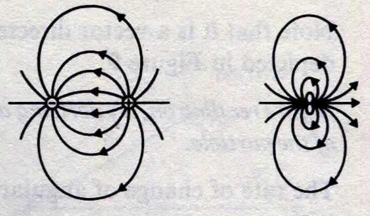



Figure 6 Lines of force showing the electric field due to (a) an electric dipole and (b) an ideal electric dipole

We have been discussing the dipole field of an electric dipole because it is easier to see how the field arises from the sum of the two simple fields due to the individual electric charges. However, it is *magnetic* dipole fields that concern us most directly in this Course. A magnetic field can be produced by current-carrying wires. In Figures 7a and b are shown the lines of force around straight wires carrying current into and out of the paper. If a wire is in the form of a loop, we get the field shown in Figure 8 (the loop is seen from the side). But, as you will readily note, this magnetic field bears a marked resemblance in form to the electric field produced by an electric dipole.

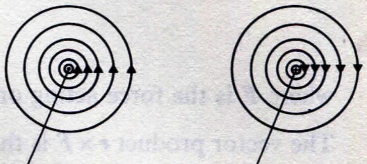



Figure 7 Lines of force showing the magnetic field due to a current passing along a wire (a) out of, and (b) into the plane of the paper.

As the current loop in Figure 8 is made smaller, a field is obtained that in the limit becomes that of an *ideal magnetic dipole*. That is, the manner in which the direction and relative magnitude of the magnetic field vary from point to point is similar to that of the field of an ideal electric dipole (Fig. 6b).

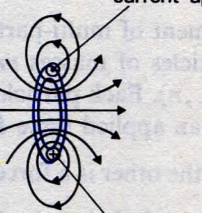



Figure 8 Lines of force showing the magnetic field due to a current-carrying loop.

If the current in the small loop is $I$ and the loop has area $A$, then it is found experimentally that the strength of the magnetic field produced by the current at a point far from the loop (the distance being large compared with the diameter of the loop) is proportional to the product $IA$. We define a vector $\boldsymbol{\mu} = IA\boldsymbol{e}$ called the *magnetic dipole moment*, where $\boldsymbol{e}$ is a unit vector in a direction normal to the plane containing the loop. The sense of $\boldsymbol{e}$ depends on the direction of the current and is given by the right-hand screw rule, for example from left to right in Figure 8. The strength and orientation of the magnetic dipole field is characterized by $\boldsymbol{\mu}$ in the same way as the strength and orientation of the electric dipole field was characterized by $Q\boldsymbol{d}$.

Some atoms generate a dipole magnetic field—hence the importance of this discussion. This does not mean that one has necessarily to picture the atom as containing electric charges going around loops within the atom (though that will indeed be the essence of our first model of the atom). The important thing is that, regardless of what one might imagine to be the mechanism producing them, some atoms do have such fields and we shall be characterizing them by their magnetic dipole moments.

As was stated earlier we shall pursue this subject further in Unit 13.

### 1.1.4 Angular momentum

Another important property possessed by atoms is *angular* momentum. How is it defined? In this Unit we consider the classical mechanical definition.

Suppose a particle of mass $m$ at position P moves with velocity $\boldsymbol{v}$. The point P is at position vector $\boldsymbol{r}$ relative to a fixed point O (see Fig. 9). The *linear* momentum, $\boldsymbol{p}$, is given by $\boldsymbol{p} = m\boldsymbol{v}$.

*The angular momentum* $\boldsymbol{L}$ *of the particle about the fixed point O is defined by the cross product**

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \\ &= m\boldsymbol{r} \times \boldsymbol{v} \qquad (6) \end{aligned}$$

*** If you are unfamiliar with cross products of vectors, you should refer to Appendix 2.**

Note that it is a vector directed out of the plane of the paper for the configuration depicted in Figure 9.

*Before reading on, try deriving an expression for the rate of change of angular momentum of the particle.*

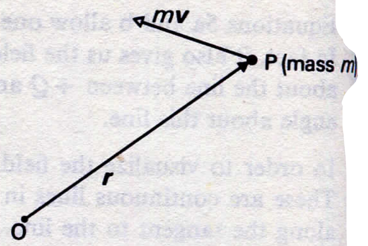



Figure 9 The definition of angular momentum.

The rate of change of angular momentum is obtained by differentiating equation 6 with respect to time:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{r}\times\boldsymbol{v}) = m\boldsymbol{v}\times\boldsymbol{v}+m\boldsymbol{r}\times\dot{\boldsymbol{v}} \qquad (7)$$

But $\boldsymbol{v}\times\boldsymbol{v} = \boldsymbol{0}$, so equation 7 becomes

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= m\boldsymbol{r}\times\dot{\boldsymbol{v}}\\ &= \boldsymbol{r}\times\boldsymbol{F}\end{aligned} \qquad (8)$$

where $\boldsymbol{F}$ is the force acting on the particle $m$.

The vector product $\boldsymbol{r}\times\boldsymbol{F}$ is the moment of the force about the fixed point O and is called the *torque* of the force about O.

*Therefore, the rate of change of angular momentum of a particle about a fixed point* O = *the applied torque about* O.

Note that the values of the angular momentum and the torque depend upon which point has been chosen as the fixed point. As long as the torque and angular momentum are referred to the same fixed point, however, the theorem holds true. Notice that we have only made some definitions; the dynamical laws are still those of Newton.

The treatment of multi-particle systems follows along similar lines. Suppose there are $n$ particles of masses $m_i$ at position vectors $\boldsymbol{r}_i$ from the fixed point O (where $i = 1, 2, \ldots, n$). Each particle can be considered to have two types of force acting on it. One is an applied force $\boldsymbol{F}_i$ generated by some agency external to the system of particles; the other is a force $\sum_{j=1}^{n}\boldsymbol{f}_{ij}$ due to the interaction of the other particles of the system on it, $\boldsymbol{f}_{ij}$ being the force exerted on the $i$th particle by the $j$th particle (the term $\boldsymbol{f}_{ii}$ being excluded). The resultant force on the $i$th particle is therefore

$$\left(\boldsymbol{F}_i+\sum_{j=1}^{n}\boldsymbol{f}_{ij}\right)$$

The total angular momentum of the system of particles is defined as the sum of the angular momenta of the individual particles:

$$\boldsymbol{L} \equiv \sum_{i=1}^{n} m_i\boldsymbol{r}_i\times\boldsymbol{v}_i$$

Taking the time derivative we get

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i\boldsymbol{r}_i\times\boldsymbol{v}_i\\ &= \sum_{i=1}^{n} m_i\boldsymbol{v}_i\times\boldsymbol{v}_i+\sum_{i=1}^{n} m_i\boldsymbol{r}_i\times\dot{\boldsymbol{v}}_i\\ &= \sum_{i=1}^{n} m_i\boldsymbol{r}_i\times\dot{\boldsymbol{v}}_i \qquad (\text{since } \boldsymbol{v}_i\times\boldsymbol{v}_i = \boldsymbol{0})\end{aligned}$$

Because the rate of change of momentum of the $i$th particle is equal to the resultant force on that particle, we have

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} &= \sum_{i=1}^{n}\left[\boldsymbol{r}_i\times\left(\boldsymbol{F}_i+\sum_{j=1}^{n}\boldsymbol{f}_{ij}\right)\right]\\ &= \sum_{i=1}^{n}\boldsymbol{r}_i\times\boldsymbol{F}_i+\sum_{i=1}^{n}\left[\boldsymbol{r}_i\times\sum_{j=1}^{n}\boldsymbol{f}_{ij}\right] \qquad (i\neq j)\end{aligned}$$

But, in the second sum on the right-hand side, there are pairs of terms $(\boldsymbol{r}_i\times\boldsymbol{f}_{ij})$ and $(\boldsymbol{r}_j\times\boldsymbol{f}_{ji})$. These are the contributions due to the mutual interaction between the

*i*th and *j*th particles. However, we know from Newton's third law that $f_{ij} = -f_{ji}$. If we assert, moreover, that these forces act along the line joining the two particles, then it is clear from Figure 10 that the two terms cancel, regardless of which point is chosen as the fixed point O. We are thus left with the result

$$\frac{dL}{dt} = \sum_{i=1}^{n} r_i \times F_i \qquad (9)$$

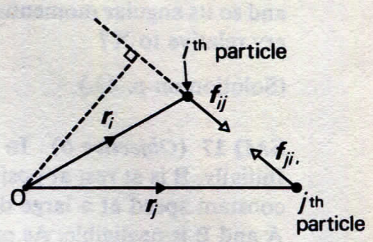


**Figure 10** The internal forces between the particles of a system cancel each other out.

We therefore reach the important conclusion that only the torques *external* to a system of particles change the net angular momentum. It is seen from the general relation 9 that if the resultant torque applied to the system (the expression on the right-hand side) is zero, then $L$ is a constant. *This is known as the law of conservation of angular momentum.*

The solar system is an example of a system that is comparatively free from external influences.* In the absence of externally applied torques, one can say immediately that, regardless of any internal torques between the planets and Sun, the overall angular momentum of the system (i.e. the sum total of the angular momenta due to the motion of the Sun and the orbiting planets and their motion as they spin on their axes) must remain constant.

The same considerations apply to an atom. An atom can possess angular momentum about its centre of mass. If it is left to itself, this angular momentum remains constant. An intriguing aspect of the angular momentum of an atom is that it can only take on certain permitted values. This is quantum-mechanical effect and we shall have more to say of this in Units 2 and 13.

A few final points: later in this Unit, and in Unit 2, you will need to recall that if a particle of mass $m$ moves with a constant speed $v$ in a circle of radius $r$, then the motion requires a force $mv^2/r$ towards the centre of the circle: the so-called centripetal force. If by any chance you have forgotten this expression, you can find a derivation of it in S 100, Unit 3.

You might also consider whether you need to revise simple kinematic formulae for uniformly accelerated motion such as

$$v^2 = v_0{}^2 + 2ax$$
$$v = v_0 + at$$
$$x = v_0 t + \tfrac{1}{2}at^2$$

where $x$ is the distance travelled, $a$ is the acceleration, $v$ and $v_0$ the final and initial velocities respectively, and $t$ is the time. You will find these also in S 100, Unit 3.

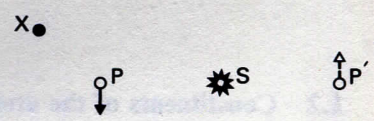


**Figure 11** SAQ 16.

So ends our quick round-up of classical mechanics. *One of the interesting features of our Course will be to see how, and in what modified form, these various classical concepts carry over into quantum mechanics.*

You should now be able to meet Objective 6. The following SAQs will help you to check whether you can.

**SAQ 15** (*Objective 6*) A planet revolves about a sun in an elliptical orbit. How does the angular momentum of the planet about the centre of the sun vary with time? (Both sun and planet are considered to behave like point particles.) **short**

(Solution on p. 43.)

**SAQ 16** (*Objective 6*)

In Figure 11, a sun S and its planet P comprise a system of particles subject to no external forces. The angular momentum of P about the arbitrary fixed point X in the diagram is not constant. (This can be seen by noting, for example, that when the planet has performed half a revolution and is in position P′, its motion is reversed **short**

* More precisely, we mean that the acceleration of the solar system in its orbit about the Galactic centre is less than the acceleration of the planets in their circumsolar orbits. Just the reverse is true for velocities. The velocity of the Earth around the Galaxy is greater than its velocity around the Sun by a factor of ten.

and so its angular momentum about X has the opposite sign.) Can S remain stationary relative to X?

(Solution on p. 43.)

**SAQ 17** (*Objective 6*) In Figure 12, A and B are two charged atomic particles. Initially, B is at rest at position b and A is at position a moving in a straight line at constant speed at a large distance from b such that the initial interaction between A and B is negligible. As particle A approaches particle B, the mutual electrostatic

**medium**

**hint on tape 1, band 2**

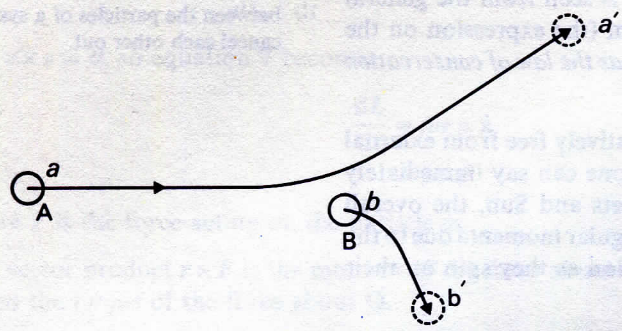



Figure 12 SAQ 17.

forces increase and cause A to be deflected from its original path. Meanwhile, B moves off along the path shown in the figure. At some later time, A is at position a′ and B is at b′ where the distance a′b′ is large and the electrostatic forces are once again insignificant. Tick any of the following statements that are true:

(i) The angular momentum of A about the fixed point b is constant throughout the motion.
(ii) The initial and final values of the angular momentum of A about the fixed point b are the same.
(iii) The angular momentum of A about the fixed point b′ is constant throughout the motion.
(iv) The initial and final values of the angular momentum of A about the fixed point b′ are the same.
(v) The initial angular momentum of A about the fixed point a′ is equal to the final angular momentum of B about the fixed point a′.

(Solution on p. 44.)

## 1.2 Constituents of the atom

### 1.2.1 The discovery of the electron

Before we can formulate even a simple model of the atom, we must first find out what it consists of. Broadly speaking, there are two kinds of atomic constituent—the electron and the nucleus. First we take a look at the experimental evidence for the existence of the electron. We need to measure both its electric charge $e$ and its mass $m$. Historically, this was done in a two-stage process in which first the ratio $e/m$ was measured, and then the value of $e$. The mass $m$ was then calculated from the values of these two quantities. The relevant experiments are described in a reading from the set book *Fundamentals of Modern Physics* by R. M. Eisberg.*

Now read Eisberg (E) from p. E 70 beginning at the second paragraph: 'The conduction of electricity...' to the end of section E 3 on p. E 75. Once again, remember to refer to the following additional notes if anything is not clear.

* R. M. Eisberg (1961) *Fundamentals of Modern Physics*, John Wiley.

## Additional notes

*p. E 71, line 6 up* footnote

Do not worry unduly if this footnote is not too clear. The statement is correct, but the underlying explanation is very complicated. In any event, the apparatus you really have to understand is that shown in Figure E 3–2, and there the anode is centrally placed and radially symmetric. Therefore, the problem does not arise because there is no net radial force on an electron passing along the axis of the system.

*p. E 72, line 1* …'In 1897 *Thomson* made accurate measurements of $e/m$, the ratio of'…

This is J. J. Thomson, not to be confused with his son, G. P. Thomson who (as you may recall from S 100, Unit 29), was one of the discoverers of the wave behaviour of the electron.

*p. E 72, line 10* …'the two plates create a force $F = eV/d$ which acts on the particles in *a*'…

The force on the electron is its charge $e$ multiplied by the potential gradient. Between parallel plates, it is found that this gradient is almost constant. It is therefore equal to the total potential difference between the plates, i.e. the voltage $V$, divided by their separation.

*p. E 72, line 14* …'emerging, their transverse deflection is $\delta = \frac{1}{2}at^2 = \frac{1}{2}(e/m)(V/d)(l/v)^2$.'…

The acceleration is given by $a = \mathrm{d}^2\delta/\mathrm{d}t^2$, so integrating twice we get $\delta = \frac{1}{2}at^2$.

*p. E 72, line 16* …'for small deflections, is *almost exactly* $2L/l$, where $L$ is the distance from'…

You will be asked to prove this for yourself in SAQ 19 on p. 24.

*p. E 73, line 6* equation (3–2)

The magnetic force on a particle of charge $e$ moving with velocity $v$ in a magnetic field $H$ is given by the expression $(ev \times H)/c$, where $c$ is the speed of light. The force is in a direction perpendicular to that of the field, as indicated by the vector cross product (see Appendix 2). You may already have come across this type of force in S 100, Unit 6 during the discussion there of the mass spectrometer.

*p. E 73, line 11* '$e/m = 5.27 \times 10^{17} esu/gm$'

Remember that if you want to know more about units, you should refer to the Section 'Units: SI units and conversion factors' in the Glossary to the Course.

*p. E 74, line 1* …'very small drops of liquid *often pick up electric charges*. If a voltage $V$'…

In the experiment, droplets are sprayed through an atomizer (hair-spray bottle) and some of these droplets are charged by static electric effects. In addition, a radioactive source can be used to ionize the air through which the droplets fall; the droplets then acquire charge as they fall.

*p. E 74, line 9* …'until it reaches *terminal velocity* because the frictional drag $F_v$ becomes'…

As the droplet falls through the air it accelerates under the influence of gravity. However, the retarding force exerted by the air increases with velocity. Quite rapidly, this force becomes equal to the gravitational force and no further acceleration occurs. The subsequent steady velocity is called the terminal velocity.

*p. E 74, line 10*... 'equal to the gravitational force *Mg*. According to Stokes' law, $F_v =$'...

Note from the wording of Objective 8 that you are not required to remember the form of Stokes' law.

*p. E 74, line 23* equation (3–5)

Once again, you are reminded to look at the note on units contained in the Glossary to the Course.

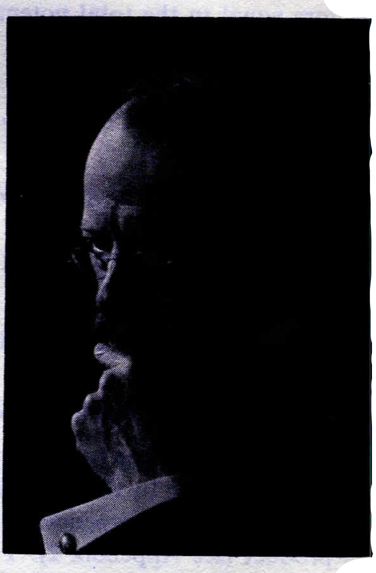
*p. E 75, line 7*... 'chemical atomic weight of hydrogen, divided by *Avogadro's number.*' ...

Avogadro's number is the number of atoms of the $^{12}C$ isotope in exactly 12 g of $^{12}C$. (See for example S 100, Unit 6.)

*p. E 75, lines 13–16*... 'In recapitulation, the electron is a particle having a negative charge'...

This paragraph summarizes the conclusions drawn by Eisberg from these specific experiments. In Unit 3, we shall see that the conclusion that the electron is a particle is not so straightforward.

The electric charge *e* and the mass *m* of an electron are not its only properties of interest. Later in the Course, you will be presented with further experimental evidence which will show, for example, that the electron also possesses an 'intrinsic' magnetic moment (i.e. one not due to any orbital motion it may possess).



J. J. Thomson (1856–1940)

You should now be able to meet Objectives 7 and 8. Incidentally, it is perhaps worth noting that in the formulation of these Objectives, and in other Objectives you will meet later in the Course, we specify a certain number of words for your answers. These are intended only as a *very* rough guide. In the case of Objectives 7 and 8 of this Unit, you will see that the numbers correspond approximately to the length of the description given in the set book. Thus we are telling you that brief descriptions of the experiments, like those given in Eisberg, are perfectly adequate for the purposes of answering examination questions; there is no need to hunt around in other text books for fuller accounts in order to pass the examination. (Mind you, as you will already appreciate, passing examinations is in itself a very limited objective. We hope that at various stages in your study of this and other courses your interest will be sufficiently aroused for you to want to seek out other books.)

To test how carefully you studied that passage from Eisberg, try the following SAQs—they will help you meet Objectives 7 and 8.

**SAQ 18** (*related to Objective 7*) With regard to Thomson's $e/m$ experiment: **long**

(i) Are the electrostatic deflection plates connected to the same voltage supply as the cathode and anode that initially accelerate the electron beam?
(ii) Would the apparatus work without a pump?
(iii) Derive the expression for the deflection on the screen produced by the electrostatic deflection plates.
(iv) How is the velocity of the electrons determined?
(v) Thomson was not content to do the experiment once only. In what ways did he vary the experimental arrangement in his repeated observations?

(Solution on p. 44.)

**SAQ 19** (*related to Objective 7*) In Eisberg's description of the Thomson experiment for the measurement of $e/m$, it states on p. E 72 that the deflection $\delta$ on emerging from the parallel metal plates is magnified by a factor of $2L/l$ by the time the electrons reach the screen. Prove this assertion. **long**

**hint on tape 1, band 2**

(Solution on p. 44.)

**SAQ 20** (*related to Objective 8*) With regard to Millikan's experiment for measuring *e*: **medium**

(i) Does the apparatus have to be evacuated?

(ii) Why was it necessary to repeat the experiment many times with different droplets?

(iii) List the quantities that need to be measured *directly*.

(Solution on p. 44.)

### 1.2.2 The discovery of the nucleus

The experiments so far described appear to show that atoms contain negatively charged electrons. However, it is also known that in general atoms have no *net* charge. There must therefore be another constituent of atoms—one that is positively charged. Moreover, in atomic terms electrons are very light; the mass of all the electrons that can be removed from an atom is less than one-thousandth of the entire atomic mass. So, whatever its detailed structure, the remaining positively charged part of an atom must be relatively massive. In order to find out what kind of structure this positively charged constituent possesses, it is useful to make a detailed study of the way such structures behave when they collide and bounce off each other.

We now turn to this type of investigation.

Read from p. E 87 to the end of section E 4.3 on p. E 92.

**Additional notes**

*p. E 87, line 10 up*...'scattering of X-rays from atoms. *These experiments will be discussed* in'...

In fact, we shall not be considering them in this Course.

*p. E 88, line 10*...'assumed to be spherical in shape with a radius *of the order of* $10^{-8}$ *cm*'...

The figure of $10^{-8}$ cm for the radius of an atom can be estimated experimentally in a variety of ways, for example, by studying the scattering of X-rays from crystals. The interference behaviour of the X-rays gives a measure of the spacing of the atoms in the crystal and hence of the size of the atoms themselves.

*p. E 88, line 17*...'equilibrium positions. Since the *electromagnetic theory predicts* that an'...

The fact that an accelerated charged body emits electromagnetic radiation will be discussed a little more in Unit 2.

*p. E 89, line 3*...'as *U and Ra*. This phenomenon will be discussed in detail later in this'...

U and Ra denote uranium and radium respectively.

Eisberg now goes on to discuss the predictions that stem from the Thomson model of the atom. There is no need for us to discuss his erroneous idea in detail. It is only one among many erroneous ideas! There is, however, some point in seeing, in general, how the size of a scattering centre affects the cross-section. The essence of these predictions is that it is very difficult to get large angular deflections of the alpha particles. This is because in Thomson's model the positively charged matter is smeared out over the whole volume of the atom and so it is impossible for the $\alpha$-particle to get really close to an intense concentration of charge capable of

exerting a sufficiently large force on it to cause a large deflection. For example, the model predicted a value $\sim 10^{-3\,500}$ for the fraction of $\alpha$-particles scattered through more than 90° in one particular experiment; the experimental value was $\sim 10^{-4}$. According to Rutherford, such a value of $10^{-4}$ could only be explained in terms of a local dense concentration of positive charge which can act as a strong scattering centre. Thus, in 1911, he was led to propose a model of the atom in which all the positive charge of the atom, and essentially all its mass, is concentrated in a small region called the *nucleus*. With this model, those alpha particles that pass near to such nuclei undergo repulsive Coulomb forces large enough to cause large deflections. This will not happen very often because the nucleus is small, but in the Thomson model it could *never* happen.

E. Rutherford (1871–1937)

In the next reading from Eisberg, an expression is derived for $N(\Phi)\,d\Phi$, the number of $\alpha$-particles scattered within the angular range $\Phi$ to $\Phi + d\Phi$. This prediction of the Rutherford model is then compared with experiment.

Before tackling this passage, note carefully the wording of Objective 10—it could save you a lot of time and trouble. We are *not* expecting you to remember the whole derivation of the Rutherford formula. In any examination question based on this Objective, we shall give you all the relevant information on the derivation up to a certain stage. You will then be required to apply your general knowledge of physics and mathematics to get from that point in the derivation to some other specified point. Indeed, whether you study this passage closely or not, you ought to be able to answer the examination question on the basis of little more than general background knowledge. However, the more thoroughly you study this passage, the *quicker* you will be able to spot the way of doing the question—and in any examination, time is precious. If you are still in any doubt about the type of assessment implied by Objective 10, take a quick look at the form of SAQs 22–26 before reading further. (Incidentally, you will be meeting this type of Objective again later in the Course wherever you have to study a lengthy derivation.)

Now read sections E 4.7 to E 4.9 inclusive, pp. E 100–108.

You are STRONGLY ADVISED to read these particular sections in close association with the additional notes provided below. It is very important that at this early stage in the Course you make yourself thoroughly familiar with the type of tutorial comment to be found in the additional notes. Should you find the comments unhelpful and trivial, then you may consider ignoring them. But for the present, you owe it to yourself to find out what help is being offered. If you develop the habit of studying the set book in conjunction with the additional notes, you could save yourself a great deal of study time. So if you have until this point skipped the additional notes—for this passage DON'T!

**Additional notes**

*p. E 100, line 23*... '*above, the scattering due to the atomic electrons* *can be ignored.* The'...

This is dealt with in detail in the second paragraph on p. E 93. There it is stated that if the mass of the alpha particle $M$ is very much greater than that of the electron $m$, then the final velocity of the electron $v_e$ can only be twice the original velocity of the alpha particle $v$. The proof is as follows:

The greatest energy transfer to the electron will occur for a head-on collision. In this case, both particles move in the same straight line. If $v'$ is the final velocity of the alpha particle, then energy conservation gives

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}Mv'^2 + \tfrac{1}{2}mv_e^2 \qquad \text{(i)}$$

Momentum conservation yields

$$Mv = Mv' + mv_e$$

i.e.

$$v' = \frac{Mv - mv_e}{M} \qquad \text{(ii)}$$

Substituting (ii) into (i), we find

$$\frac{1}{2}Mv^2 = \frac{1}{2}M\left(\frac{M^2v^2 - 2Mmvv_e + m^2v_e^2}{M^2}\right) + \frac{1}{2}mv_e^2$$

$$0 = -mvv_e + \frac{1}{2}\frac{m^2v_e^2}{M} + \frac{1}{2}mv_e^2$$

The second term on the right is negligible compared with the third because $M \gg m$, thus

$$v_e = 2v$$

In order for the alpha particle to be deflected sideways, it must strike a glancing blow, in which case

$$v_e < 2v$$

It should be emphasized that this treatment assumes that the collision is an elastic one, i.e. the nucleus is not left in an excited energy state, and no electromagnetic radiation is emitted.

*p. E 100, line 3 up . . . '$v/c \sim 1/20$.'*

Corrections due to special relativity theory are of the order $(v/c)^2$. In the case we are considering, they would affect results by 1 part in 400 and so can be ignored.

*p. E 101, line 16 . . .* 'Consequently its angular momentum $Mr^2(\mathrm{d}\theta/\mathrm{d}t)$ has *the constant value L.' . . .*

We are here talking about the angular momentum about the position of the nucleus. As the force is along the radial direction, there is no torque about this point and hence no change in angular momentum.

*p. E 102, line 5* equation (4–9)

If the alpha particle were moving directly towards the nucleus along a radius, there would only be the first term on the right-hand side of the equation, and this would give the familiar form of Newton's second law. However, the alpha particle also has a component of velocity at right angles to the radius. In other words, the radius vector to the particle is rotating about the nucleus, rather as though the particle were in orbit. There will, therefore, be a need for a centripetal force $\dfrac{mV^2}{r}$. But because the velocity $V$ at any instant can be written $V = r\,\mathrm{d}\theta/\mathrm{d}t$, we get for the second term on the right-hand side of equation E (4–9), $mr(\mathrm{d}\theta/\mathrm{d}t)^2$.

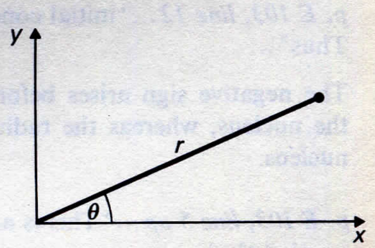



**Figure 13** **Definition of polar coordinates.**

Alternatively, we can look at it in the following way: In Figure 13

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The acceleration component in the $r$ direction is $\ddot{x}\cos\theta + \ddot{y}\sin\theta$, where

$$\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2, \quad \text{and} \quad \ddot{y} = \mathrm{d}^2y/\mathrm{d}t^2$$

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\sin\theta\dot{\theta} - r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta}$$

Similarly,

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\cos\theta\dot{\theta} - r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta}$$

Therefore

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r}(\cos^2\theta + \sin^2\theta) - r\dot{\theta}^2(\sin^2\theta + \cos^2\theta),$$
$$= \ddot{r} - r\dot{\theta}^2$$

*p. E 102, line 5* equation (4–9)

In SI units, the term $zZe^2/r^2$ should be replaced by $zZe^2/4\pi\epsilon_0 r^2$. In general, we replace $zZe^2$ wherever it appears by $zZe^2/4\pi\epsilon_0$.

*p. E 102, line 8* ... '(radially directed) *centrifugal acceleration*. To effect the fastest solution' ...

'Centrifugal' acceleration ought to read centripetal. See Glossary definition of CENTRIFUGAL FORCE.

*p. E 102, line 13* $'\dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt} = \dfrac{dr}{du}\dfrac{du}{d\theta}\dfrac{d\theta}{dt}'$

The chain rule for derivatives is being used here. You may be a little puzzled about the factor $du/d\theta$; after all, $u$ and $\theta$ are supposed to be the coordinates, so how can $u$ be a function of $\theta$? The answer is that both $u$ and $\theta$ are functions of time:

$$u : t \longmapsto u(t)$$

$$\theta : t \longmapsto \theta(t)$$

$t$ is of no use in the final result—only $u$ and $\theta$ can be measured. Therefore, we eliminate $t$ and write $u$ in terms of $\theta$:

$$u : \theta \longmapsto u(\theta)$$

*p. E 102, line 14* $'\dfrac{dr}{dt} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}\dfrac{Lu^2}{M} = -\dfrac{L}{M}\dfrac{du}{d\theta}'$

We are substituting for $d\theta/dt$ from equation E (4–8).

*p. E 102, line 4 up* ... 'potential energy $zZe^2/D$ is equal to the kinetic energy $\frac{1}{2}Mv^2$; at this' ...

During a head-on collision, the particle velocity is reduced owing to the repulsive action of the nucleus, and finally becomes zero at the point of closest approach. Since the total energy remains constant, the energy at the point of closest approach (all potential) equals the initial energy, all of which was kinetic.

*p. E 103, line 12* ... 'initial conditions: $\theta \to 0$ as $r \to \infty$, and $dr/dt \to -v$ as $r \to \infty$. Thus' ...

The negative sign arises before $v$ because the initial velocity is directed *towards* the nucleus, whereas the radius vector to the particle is directed *away* from the nucleus.

*p. E 103, line 5 up* ... 'This is an equation of a hyperbola in polar coordinates. We see that the' ...

There is no need to remember that this is one of the forms in which the equation of a hyperbola can be expressed. The equation will be quoted to you in any assessment question that relates to it. Students of MST 282 in particular should note that the origin of the polar coordinates is not the focus of the hyperbola; this accounts for the equation being somewhat different from that derived in MST 282.

*p. E 104, line 9* $'\dfrac{2b}{D} = \dfrac{1-\cos\theta'}{\sin\theta'} = \tan\theta'/2'$

Remember the trigonometric relations $\sin\theta' = 2\sin(\theta'/2)\cos(\theta'/2)$

$$\cos\theta' = 1 - 2\sin^2(\theta'/2)$$

*p. E 104, line 18* ... '*polar angle is equal to* $\theta'/2 = (\pi-\theta)/2$. Evaluating equation (4–15)' ...

This value for the polar angle can be proved by using the standard calculus method for finding the maximum value of $1/r$ in the equation E (4–15).

*p. E 104, line 13 up* equation (4–17)

The derivation of this equation is left as an exercise to be worked out in SAQ 23.

*p. E 104, line 12 up* ... 'It is easy to verify that $R \rightsquigarrow D$ as $\phi \rightsquigarrow \pi$, and that $R \rightsquigarrow b$ as $\phi \rightsquigarrow 0$, as' ...

To verify that $R \rightsquigarrow b$, substitute for $D$ from equation E (4–16) before letting $\phi \rightsquigarrow 0$. Note also from E (4–16), that $b$, and hence $R$, $\rightsquigarrow \infty$ as $\phi \rightsquigarrow 0$. In other words, the particle is not deflected because its initial direction does not take it close to the nucleus.

*p. E 105, line 6* ... 'to *the total area obscured by the rings*, as seen by the incident alpha' ...

How important is the possibility that the rings might overlap? As you will see later (p. E 106), experiments performed by Geiger and Marsden covered an angular range 5° to 150°. The largest value of interest for the impact parameter $b$ and hence for the ring radius can be calculated as follows:

$$b = \frac{D}{2}\cot(\phi/2)$$

From p. E 107, we find that

$$D \sim 1.7 \times 10^{-12} \text{ cm for copper}$$

Thus $\quad b = 1.9 \times 10^{-11}$ cm

for $\quad \phi = 5°$

The interatomic distance is of the order of $10^{-8}$ cm. Thus the chance of an alpha particle striking an atom within a distance $b$ of the centre of that atom is the ratio of the areas:

$$\frac{\pi \times (2.10^{-11})^2}{\pi \times (10^{-8})^2} = 4 \times 10^{-6}$$

Geiger and Marsden used very thin foils, of the order of $10^{-4}$ cm thick. This means each alpha particle passed through about $10^4$ atoms. Thus the overall chance of an alpha particle passing within a distance $b$ of the centre of some atom was about $4 \times 10^{-6} \times 10^4 \approx 10^{-1}$. So the chance of the particle passing through *two* atoms within a distance $b$ was just about negligible.

*p. E 105, line 10 up* $\quad$ '$\mathrm{d}b = -\dfrac{D}{2}\dfrac{\frac{1}{2}\,\mathrm{d}\phi}{\sin^2(\phi/2)}$'

The factor of $\frac{1}{2}$ in the numerator comes from $\mathrm{d}(\phi/2) = \frac{1}{2}\mathrm{d}\phi$

*p. E 105, line 8 up* $\quad$ '$b\,\mathrm{d}b = -\dfrac{D^2}{8}\dfrac{\cos(\phi/2)\,\mathrm{d}\phi}{\sin^3(\phi/2)} = -\dfrac{D^2}{16}\dfrac{\sin\phi\,\mathrm{d}\phi}{\sin^4(\phi/2)}$'

The last step is obtained by multiplying the numerator and denominator by $\sin(\phi/2)$ and putting $2\sin(\phi/2)\cos(\phi/2) = \sin\phi$

*p. E 105, line 4 up* ... 'be scattered in the angular range $\phi$ to $\phi + \mathrm{d}\phi$ (*The minus sign compensates*' ...

If $\mathrm{d}\phi$ is to be positive, $\mathrm{d}b$ will be negative (as $\phi$ increases, $b$ decreases). In order to make the probability a positive quantity, the expression for it must be $-P(b)\,\mathrm{d}b$ instead of $P(b)\,\mathrm{d}b$ as stated on line 5 of this page.

*p. E 105, line 2 up* ... '*notation employed in equation (4–5)*, this is' ...

The notation referred to is simply that $\mathcal{N}(\Phi)\,\mathrm{d}\Phi$ is the number of alpha particles scattered within the angular range $\Phi$ to $\Phi + \mathrm{d}\Phi$ and $\mathcal{N}$ is the total number of alpha particles traversing the foil.

*p. E 106, line 1* '... Evaluating $D$, we have' ...

This is done using equation E (4–12).

*p. E 106, line 12 ...* 'angle scattering angular distribution (*equation 4–5*).'.

This is just a reference to the predictions of the Thomson model.

*p. E 106, line 21 ...* 'for a range of *thickness of about 10* for all the elements investigated.'

The largest thickness used was 10 times the smallest used.

You should now be able to meet all the Objectives of the Unit.

**SAQ 21** (*Related to Objective 9*) With regard to the Rutherford scattering experiment: **short**

(i) How were the alpha particles collimated?
(ii) Why was it necessary to evacuate the apparatus?
(iii) How were the alpha particles detected?

(Solution on p. 45.)

**SAQ 22** (*Objective 10*). In Rutherford's theory of alpha particle scattering: **short**

(i) Does the angular momentum of the alpha particle about the position of the fixed nucleus remain the same at all times during the collision?
(ii) Does the kinetic energy of the alpha particle remain the same at all times?
(iii) Is it important to take into account the energy loss of the alpha particles in traversing the foil?

(Solution on p. 45.)

**SAQ 23** (*Objective 10*). Derive equation E (4–17) on p. E 104 from equations E (4–15) and E (4–16). **long**

(Solution on p. 45.)

**SAQ 24** (*Objective 10*). Refer to Figure E 4–10 on p. E 101 for the notation, but do not refer to the text before you have attempted the questions: **long**

Applying Newton's second law to the radial component of motion we have **hint on tape 1, band 1**

$$\frac{zZe^2}{4\pi\epsilon_0 r^2} = M\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] \qquad (1)$$

(i) How do the three terms (one on the left and two on the right) arise?
After some manipulation, equation 1 above can be expressed in the form

$$\frac{d^2u}{d\theta^2} + u = -D/2b^2 \qquad (2)$$

where $u \equiv 1/r$ and $D$ is a constant.

(ii) What is the general solution of equation 2?
(iii) What are the initial conditions in this problem?
(iv) Prove that equation 2 leads to an equation of a hyperbola. You may assume that the general equation of a hyperbola has the form:

$$\frac{1}{r} = A\sin\theta + B(\cos\theta - 1)$$

where $A$ and $B$ are constants.

(Solution on p. 45.)

**SAQ 25** (*Objective 10*). Refer to Figure E 4–10 on p. E 101 for the notation, but do not refer to the text before you have attempted the question: **long**

Given that the scattering angle $\phi$ is related to the impact parameter $b$ by the equation **hint on tape 1, band 1**

$$\cot(\phi/2) = 2b/D$$

prove that for $\mathcal{N}$ alpha particles incident on a foil of thickness $t$ and containing $\rho$

nuclei per unit volume, the number scattered through angles $\Phi$ to $\Phi+d\Phi$ is

$$\frac{\pi}{8}\mathcal{N}\rho t D^2 \frac{\sin\Phi\, d\Phi}{\sin^4(\Phi/2)}$$

(For the solution refer to Eisberg's account, pp. E 104–6.)

**SAQ 26** (*Objective 10*). The expression for the number of alpha particles scattered through angles $\Phi$ to $\Phi+d\Phi$, given on p. E 106, approaches infinity as $\Phi \to 0$. Why does this not make nonsense of the Rutherford theory? **short hint on tape 1, band 2**

(Solution on p. 45.)

## 1.3 Summary of the Unit

*We began by reviewing various aspects of classical mechanics*:

1 A scalar field is a function that associates a scalar with each point in a given region of space. The gradient of a scalar field $\phi\,(x, y, z)$ at a point is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i}+\frac{\partial\phi}{\partial y}\boldsymbol{j}+\frac{\partial\phi}{\partial z}\boldsymbol{k}$$

2 A vector field is a function that associates a vector (magnitude *and* direction) with each point in a given region of space.

3 Some forces can be expressed as the negative of the gradient of a scalar called potential energy $V(x, y, z)$ thus

$$\boldsymbol{F}(x, y, z) = -\nabla V(x, y, z)$$

If such a $V$ exists, this expression is a definition of potential energy, except for the possible addition of a constant. Force fields that have associated potential energy fields are examples of conservative forces.

4 A conservative force field is one that can be represented as the negative of the gradient of a scalar field, and does not change with time. For a particle moving in such a force field, the sum of the potential and kinetic energies is independent of time and position. The non-relativistic kinetic energy is defined as $\frac{1}{2}mv^2$ where $m$ is the mass and $v$ the velocity.

5 The basic programme of mechanics (quantum as well as classical) is two-fold: (i) the specification of an instantaneous 'state' of the mechanical system; and (ii) the study of how this 'state' evolves with time.

6 The instantaneous state of a particle in classical mechanics is specified by giving the values of its position and momentum at some 'initial' time. For $n$ particles it is the set $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n; \boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$.

7 The subsequent evolution of the state can be investigated in the Newtonian formulation by integrating Newton's second law twice—the two constants of integration being determined from the initial values of $x$ and $p$.

8 Alternatively, the second-order differential equation representing Newton's second law can be replaced by two coupled first-order differential equations (equation G (3–10a) and G (3–10b) p. G 37) in the Hamiltonian formulation.

The two formulations are equivalent.

9 These coupled first-order differential equations involve the Hamiltonian function $H(x, p)$, which we choose to define as the total energy expressed as a function of the state variables $x$ and $p$.

10 An electric dipole consists of two equal and opposite charges, $+Q$ and $-Q$, placed a distance $d$ apart. An ideal electric dipole is one for which $d$ is vanishingly small compared with the distance to any point where the field is considered, but the product $Qd$ does not vanish.

11 A magnetic dipole field is a magnetic field exhibiting the same configuration as the electric field produced by an electric dipole. A small loop carrying a current $I$ and having an area $A$ produces a magnetic dipole field. It is characterized by the magnetic dipole moment $\boldsymbol{\mu}$ defined by $\boldsymbol{\mu} = IA\boldsymbol{e}$, where $\boldsymbol{e}$ is a unit vector normal to the plane containing the loop, and in a direction given by the right-hand screw rule. If the loop is made vanishingly small compared with the distance at which the field is observed, the field assumes the same form as that of an ideal electric dipole, and is called an ideal magnetic dipole field.

12 For a system of particles of mass $m_i$, velocity $\boldsymbol{v}_i$, at position vector $\boldsymbol{r}_i$ from a fixed point O (where $i = 1, 2, \ldots, n$), the total angular momentum $\boldsymbol{L}$ is defined by

$$\boldsymbol{L} \equiv \sum_{i=1}^{n} m_i \boldsymbol{r}_i \times \boldsymbol{v}_i$$

13 The rate of change of $\boldsymbol{L}$ is equal to the net torque applied externally to the system:

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i$$

*Having completed the brief review of classical mechanics, we turned to the experimental evidence concerning the nature of the constituents of atoms:*

14 A description was given of J. J. Thomson's experiment for measuring $(e/m)$ for an electron (p. E 72).

15 A description was given of Millikan's experiment for measuring the electronic charge $e$ (p. E 73). Thus, knowing $e/m$ and $e$, one can deduce $m$. It is found that the electron has negative charge, the magnitude of which is equal to the magnitude of the charge on a singly ionized atom. Furthermore, the electron has a mass smaller than that of a proton (i.e. an ionized hydrogen atom) by a factor 1 836.

16 A description was given of Rutherford's experiment on the scattering of $\alpha$-particles by nuclei (p. E 91).

17 The Rutherford scattering formula was derived (equation E (4–18) on p. E 106). The fact that the formula is obeyed rather well experimentally is evidence that the massive positively charged constituent of the atom is confined to a small region of space. It is called the nucleus.

*Having described the experimental evidence for the existence of the electron and nucleus, we are now ready to consider how these constituents can be put together to form an atom; this is the subject of the next Unit.*

**Appendix 1**

# Notation and terminology

This appendix reproduces material from MST 282. Students of M 201 are advised to study it before embarking on the Course proper.

## A.1.0 Introduction

In M 100 and MST 281 we concentrated on the basic mathematical ideas. There we were extremely careful in our notation; we distinguished, for example, between functions and the images of elements under them, between arrows and geometric vectors, and we expect you to be clear about these distinctions. In this Course, we are not interested in pure mathematics for its own sake, but in how it can model the physical world.

Some of the major developments in mathematics have coincided with the introduction of an adaptable notation. Notations, like the $\sum$ sign for summation and $n!$, are simply mathematical shorthand for much longer statements. Other notations such as $\lim_{x \to 0}$ remind us instantly of the processes which they represent. For the calculus, we shall use the Leibniz notation (which was introduced briefly in M 100) because it usually makes the manipulation of derivatives and integrals considerably easier than with the function notation.

In applied mathematics, we are not so much concerned with the foundations and structure of mathematics as with the modelling of physical situations and solving the consequent mathematical problems. Thus we shall often find it worth while to gain flexibility and physical clarity by adulterating the mathematical notation. If in doubt, we can repeat any piece of work using more precise notation to clarify points or resolve arguments.

## A.1.1 Functions and variables

The essence of the Leibniz notation is that it concentrates on images under functions rather than on the functions themselves. In M 100 and MST 281 we required three things to specify a function:

(i) a set of elements $A$ called the *domain*;
(ii) a set of elements $B$ called the *codomain*;
(iii) a rule which assigns one (and only one) element of $B$ to *each* element of $A$.

If we let $f$ be the function, then we write

$$f: A \longmapsto B$$

to indicate that $f$ has domain $A$ and codomain $B$. If the *image set*, $f(A)$, is $B$ (and not a proper subset of $B$), then we write

$$f: A \longmapsto B$$

In a particular element $b \in B$ is assigned to the element $a \in A$, then we write

$$b = f(a)$$

or, alternatively, we use the notation

$$f: a \longmapsto b$$

and say that $a$ is mapped to $b$ under the function $f$. We know that the latter notation can be used to define a function explicitly; for example,

$$f: x \longmapsto x^2 \qquad (x \in R)$$

is the function with domain $R$ (the set of real numbers) such that each element $x \in R$ is mapped to its square.

The symbol $x$ used in this definition of the function $f$ is called a variable in the domain of $f$. If we let $y$ be the image of $x$ under a particular function $f$, so that $y = f(x)$, then $y$ is the corresponding variable in the codomain. Sometimes we call $x$ and $y$ the independent and dependent variables respectively: an *independent* variable is an element in the domain, and the *dependent* variable is the corresponding element in the codomain.

Often we wish to discuss the behaviour of the variables, and it is inconvenient to have to name the function which relates them.

For example, the functions

$$x \longmapsto \sin x \qquad (x \in R)$$

and

$$x \longmapsto x^2 \qquad (x \in R)$$

have the natural names 'sine' and 'square,' but the function

$$x \longmapsto \sin(x^2 + \cos x) \qquad (x \in R)$$

has no natural name. In the latter case, we often speak loosely of the 'function'

$$\sin(x^2 + \cos x)$$

If we write

$$y = \sin(x^2 + \cos x)$$

to specify the relation between two variables $x$ and $y$, then we often say (again imprecisely) '$y$ is a function of $x$,' and we indicate this situation by writing $y(x)$ instead of $y$, to emphasize the dependence of the variable $y$ on the variable $x$.

Really this is the nub of our notational adulteration, for, using the 'image' notation $y(x)$, we *imply* that $y$ is a function and not a variable. The context usually makes the meaning clear.

The reason for concentrating on 'variable' notation appears thus far to be simply a matter of abbreviation, but there are other reasons. For example, suppose that we are asked to determine the proportions of a cylindrical can (with lid) which holds the greatest volume for a given surface area.

If we choose $r$ to be the radius of the can, $h$ to be its height, $V$ to be the volume and $A$ to be the surface area, then we have relationships between the variables $r$, $h$, $V$ and $A$. We can now write down the equations

$$V = \pi r^2 h \tag{1}$$

and

$$A = 2\pi r h + 2\pi r^2 \tag{2}$$

without referring explicitly to any of the functional relationships between the quantities.

The following question is intended to revise the concept of function, and to illustrate that the function notation can be cumbersome in some situations.

**SAQ A1.1**

One function which arises from the pair of equations relating $A$, $V$, $r$, $h$ is **medium**

$$f_1 : (r, h) \longmapsto \pi r^2 h \qquad ((r, h) \in R_0^+ \times R_0^+)$$

we have

$$V = f_1(r, h)$$

Write down four more functions which can arise from rearrangements of the equations 1 and 2.

(Solution on p. 46.)

Leaving our rigorous notation does have its dangers, of course. Using the variable notation, we have no indication of the domains of the functions involved, and we need to use care. For example, in one situation we might wish to emphasize that velocity $V$ is dependent on time $t$ and write $V(t)$ and in another situation that velocity $V$ is dependent on displacement $x$ and write $V(x)$. These statements mean that there are two distinct functions

$$f : t \longmapsto V$$

$$g : x \longmapsto V$$

but in *most* problems in applied mathematics the introduction of two distinct functions is not necessary: we simply use the variable $V$.

**SAQ A1.2** **medium**

Put each of the following into a more precise form:

(i) the 'function' $1+\sin(x^2+2)$

(ii) the 'function' $\dfrac{x^2+1}{x-1}$

(iii) velocity $v$ is a function of displacement $x$, which itself is a function of time $t$; hence velocity is a function of time.

(Solution on p. 46.)

**SAQ A1.3** **short**

We are adopting a new notation because...

A the function notation cannot cope with physical relationships
B the function notation tends to be unwieldy in practice
C the variable notation aids physical clarity

Which is/are correct?

(Solution on p. 46.)

**SAQ A1.4** **short**

$X(t)$ means:

A $X$ multiplied by $t$
B the image of an element $t$, where $t$ lies in the domain of a function $X$
C a variable $X$ is dependent on a variable $t$

Which is/are correct?

(Solution on p. 46.)

## A.1.2 Calculus

In M 100 and MST 281 we chose an approach to calculus which was intended to give clarity to the underlying concepts. That approach was different from the way in which the subject was first formulated by Newton and Leibniz, and it is the latter's approach which has several advantages in giving insight in applied mathematical problems. We plan to use the Leibniz notation predominantly in this Course. In this notation

$\dfrac{dv}{dx}$ represents the derivative of the variable $v$ at $x$

and $\int v\,dx$ represents an indefinite integral of the variable $v$ expressed in terms of the variable $x$.

In M 100 and MST 281 we defined the derived function $f'$ of a given function $f$ in the following way.

We let

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h},$$

provided the limit exists. This defines the function $f'$ with domain the set of all values of $x$ for which the limit exists. If we wish to emphasize the operational aspect of differentiation, we use $\mathrm{D}f$ or $f'$ for the derived function of $f$. If

$$y = f(x)$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$$

The notation $f'$ for the derived function was introduced by Lagrange in the eighteenth century. If $y$ denotes the variable in the codomain of $f$, it is quite common in textbooks to find $y'$ as an abbreviation for the derivative $f'(x)$*. Similarly, the higher derivatives $f''(x), \ldots, f^{(n)}(x)$ are abbreviated to $y'', \ldots, y^{(n)}$. (It is cumbersome to use dashes after the second derivative, and $y^{(3)}$ is often written in preference to $y'''$, and so on.)

The notation introduced by Lagrange is very similar to that used by Newton, who wrote $\dot{y}$ and $\ddot{y}$ instead of $y'$ and $y''$. In modern texts, the dot notation is reserved almost exclusively for derivatives of functions of *time*, particularly in books on applied mathematics.

We compare the function and variable approaches in the following elementary example involving differentiation.

*Example*

If the displacement $y$ of a particle at time $t$ is given by

$$y = \sin t$$

find the velocity and acceleration at time $t$.

*Solution*

Firstly, we can say that

$$f: t \longmapsto \sin t \qquad (t \in R_0^+)$$

determines the displacement, and from M 100 we know that the velocity and acceleration at time $t$ are respectively

$$f'(t) = \cos t \quad \text{and} \quad f''(t) = -\sin t$$

Secondly, to avoid introducing the function $f$, we could write

$$\dot{y} = \cos t \quad \text{and} \quad \ddot{y} = -\sin t$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\sin t$$

Returning now to integration, in M 100 and MST 281 we showed that the integration process defines a mapping of functions to the real numbers in the case of the definite integral, and it defines a mapping of functions to primitive functions in the case of the indefinite integral. We distinguished the two cases by writing

$$\int_a^b f \quad \text{and} \quad \int f$$

for the definite integral and the indefinite integral respectively. Now we intend to work with variables, and we shall find it more convenient to use the respective notations

$$\int_a^b f(x)\,\mathrm{d}x \quad \text{and} \quad \int f(x)\,\mathrm{d}x$$

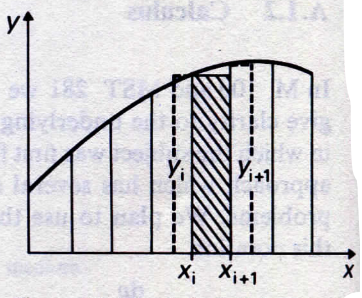


**Figure 14** The definite integral.

* This is a similar adulteration of the notation to the one we met before—that being the dual use of a symbol, $y$ in this case, to represent both a function and a variable.

or

$$\int_a^b y\,dx \quad \text{and} \quad \int y\,dx$$

We do this for two reasons. Firstly, because in the definite integral the notation reminds us that it represents the limit of a sum of the type

$$\sum_i y_i(x_{i+1}-x_i) = \sum_i y_i \Delta x_i$$

over an appropriate interval.

Secondly, because the notation enables us to perform more easily some of the necessary operations, such as substitution, needed to determine a definite or indefinite integral.

### A.1.3 Vectors

In M 100 a vector was defined as an element of a vector space; M 100 Unit 22, *Linear Algebra I* introduced the term by discussing the set of geometric vectors which, with the operations of addition of geometric vectors and multiplication of a geometric vector by a real number, forms a vector space. This particular vector space is important in mechanics, and for this reason applied mathematicians often abbreviate *geometric vector* to *vector*.

There is a second use of the word *vector* which arises in applied mathematics. Often we are dealing with quantities like velocity, acceleration and force, each of which has an associated magnitude and direction. These are often referred to as vector quantities although this term is frequently abbreviated to *vectors* in books.

The first step in forming a mathematical model of a physical problem involving vector quantities is to test whether vector quantities can be adequately modelled by geometric vectors, or representatives of geometric vectors (*arrows*). In other words, to test whether the vector quantities have the property of combining in the appropriate way, so that they can be modelled by geometric vectors.

There are various notations for vectors; for example, $\vec{a}$ or $\underline{a}$ as used in M 100 and MST 281, or $\boldsymbol{a}$ as in the text for this Course. In print, it is convenient to use bold-face type $\boldsymbol{a}$ for vectors, and to use italic print for scalars. On the television programmes we use bold-face type for vectors.

We shall also frequently use the word 'vector,' or the symbol for a vector, to stand for 'the representative of a geometric vector at a point (i.e., an arrow), which models a physical quantity.' This is not as disastrous as it might seem, because in almost all circumstances its meaning is clear from the context, and it is certainly an aid to brevity. For example, instead of writing:

> the geometric vector $\boldsymbol{F}$ whose representative at a point P is a model of the force $\boldsymbol{F}$ which is acting at P.

we write simply

> the force $\boldsymbol{F}$ at P.

Strictly speaking the force $\boldsymbol{F}$ is not a geometric vector, but it can be *represented* by the geometric vector $\boldsymbol{F}$, provided that physical forces combine in a way which the addition of geometric vectors would predict.

### A.1.4 Summary

We shall use the Leibniz notation for derivatives and integrals in this course. In situations where the context is clear, we shall emphasize the dependence of a variable $x$ on a variable $t$ by writing $x(t)$. This is a dual use of a symbol, to represent a

variable and a function, and as such it is an abuse of notation introduced in M 100 and MST 281.

We shall also use the word 'vector' to mean a representative of a geometric vector.

## Appendix 2

## Cross products of vectors

This appendix reviews material that is treated in MST 282, but not in M 201.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are any two vectors in a three-dimensional Euclidean space and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are an orthogonal basis in this space, then we have the decomposition

$$\boldsymbol{u} = u_1\boldsymbol{i}+u_2\boldsymbol{j}+u_3\boldsymbol{k}$$
$$\boldsymbol{v} = v_1\boldsymbol{i}+v_2\boldsymbol{j}+v_3\boldsymbol{k}$$

The *cross product* $\boldsymbol{u}\times\boldsymbol{v}$ is a third vector, defined as

$$\boldsymbol{u}\times\boldsymbol{v} = (u_2v_3-u_3v_2)\boldsymbol{i}+(u_3v_1-u_1v_3)\boldsymbol{j}+(u_1v_2-u_2v_1)\boldsymbol{k}$$

It is a vector at right angles to both $\boldsymbol{u}$ and $\boldsymbol{v}$, of length $|\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta$ where $|\boldsymbol{u}|, |\boldsymbol{v}|$ are the lengths of $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\theta$ is the angle between them. The sense is given by the *right-hand screw rule*, that a rotation from $\boldsymbol{u}$ to $\boldsymbol{v}$ through less than 180° would cause a screw to advance along $\boldsymbol{u}\times\boldsymbol{v}$ in the positive direction.

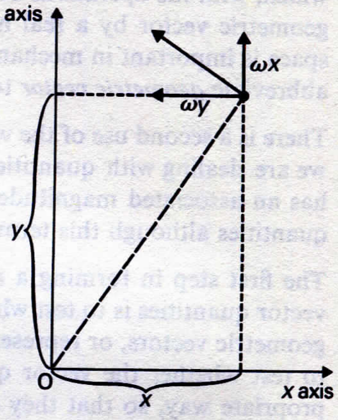


Figure 15 Cross products of vectors.

As an example, consider a body rotating about the $x$-axis with an angular rate of rotation $\omega$. At the point $(x,y)$ in Figure 15 the velocity has an $x$ component $-\omega y$ and a $y$ component $\omega x$, so it is a vector field given by

$$\boldsymbol{v}(x,y,z) = -\omega y\boldsymbol{i}+\omega x\boldsymbol{j}$$

Using cross products, this can be written more concisely as

$$\boldsymbol{v} = \omega\boldsymbol{k}\times\boldsymbol{r}$$

More generally, if the axis of rotation passes through the origin in the direction of an arbitrary unit vector $\boldsymbol{e}$ (instead of the direction of $\boldsymbol{k}$), then the velocity field is

$$\boldsymbol{v} = \omega\boldsymbol{e}\times\boldsymbol{r}$$

or

$$\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{r}$$

where $\boldsymbol{\omega}$ ($\equiv\omega\boldsymbol{e}$) is defined as an axial vector. This can be seen geometrically in Figure 16.

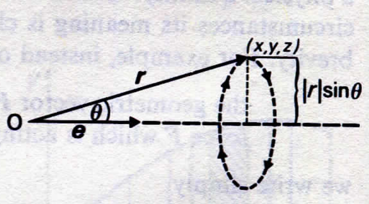


Figure 16 Cross products of vectors.

The point $(x,y,z)$ is at a distance $|\boldsymbol{r}|\sin\theta$ from the axis and therefore moves at a speed $\omega|\boldsymbol{r}|\sin\theta$ in a direction perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{e}$ (that is to $\boldsymbol{r}$ and $\boldsymbol{\omega}$). The velocity therefore has the same magnitude and direction as the vector $\boldsymbol{\omega}\times\boldsymbol{r}$, so it is either $\boldsymbol{\omega}\times\boldsymbol{r}$ or $-\boldsymbol{\omega}\times\boldsymbol{r}$. By convention we choose the sign of $\boldsymbol{\omega}$ so that a right-handed screw attached to the rotating body would advance in the positive direction, and this gives $+\boldsymbol{\omega}\times\boldsymbol{r}$ for the velocity field, as stated above.

The algebra of cross products is rather special. First of all, like matrix multiplication, it is not commutative; in fact it is anti-commutative:

$$\boldsymbol{u}\times\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{u}$$

But unlike matrix multiplication it is also non-associative:

$$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{\omega}) \neq (\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{\omega}$$

(try the example $\boldsymbol{u} = \boldsymbol{i}$, $\boldsymbol{v} = \boldsymbol{j}$, $\boldsymbol{\omega} = \boldsymbol{j}$ for yourself).

It is, however, distributive, so that

$$\boldsymbol{u}\times(\boldsymbol{v}+\boldsymbol{\omega}) = (\boldsymbol{u}\times\boldsymbol{v})+(\boldsymbol{u}\times\boldsymbol{\omega})$$

and

$$(\boldsymbol{u}+\boldsymbol{v})\times\boldsymbol{\omega}=(\boldsymbol{u}\times\boldsymbol{\omega})+(\boldsymbol{v}\times\boldsymbol{\omega}).$$

It is also linear in each factor, so that

$$\boldsymbol{u}\times(c\boldsymbol{v})=(c\boldsymbol{u})\times\boldsymbol{v}=c(\boldsymbol{u}\times\boldsymbol{v})$$

where $c$ is any number.

The cross product of any vector with itself is the zero vector

$$\boldsymbol{u}\times\boldsymbol{u}=\boldsymbol{0}$$

and if two vectors have zero cross product, they must be proportional:

$$\boldsymbol{u}\times\boldsymbol{v}=\boldsymbol{0} \text{ implies } a\boldsymbol{u}=b\boldsymbol{v} \text{ for some real numbers } a, b.$$

The following calculation illustrates some of these properties:

$$\begin{aligned}(2\boldsymbol{i}+3\boldsymbol{j})\times(4\boldsymbol{i}+6\boldsymbol{j}) &= (2\boldsymbol{i}\times4\boldsymbol{i})+(3\boldsymbol{j}\times4\boldsymbol{i})+(2\boldsymbol{i}\times6\boldsymbol{j})+(3\boldsymbol{j}\times6\boldsymbol{j}) && \text{(distributive)}\\ &= 8\boldsymbol{i}\times\boldsymbol{i}+12\boldsymbol{j}\times\boldsymbol{i}+12\boldsymbol{i}\times\boldsymbol{j}+18\boldsymbol{j}\times\boldsymbol{j} && \text{(linear)}\\ &= \boldsymbol{0}-12\boldsymbol{k}+12\boldsymbol{k}+\boldsymbol{0}\\ &= \boldsymbol{0}\end{aligned}$$

from which we conclude that $2\boldsymbol{i}+3\boldsymbol{j}$ and $4\boldsymbol{i}+6\boldsymbol{j}$ are proportional, and in fact one is twice the other.

### SAQ A2.1

short

Calculate $\boldsymbol{i}\times\boldsymbol{i}$, $\boldsymbol{i}\times\boldsymbol{j}$, $\boldsymbol{i}\times\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{i}$, etc. and hence complete a multiplication table as shown:

| | $\boldsymbol{i}$ | $\boldsymbol{j}$ | $\boldsymbol{k}$ ← second factor |
|---|---|---|---|
| $\boldsymbol{i}$ | $\boldsymbol{0}$ | $\boldsymbol{k}$ | |
| $\boldsymbol{j}$ | | | |
| $\boldsymbol{k}$ | | | |
| ↑ first factor | | | |

(Solution on p. 46.)

### SAQ A2.2

long

Use the multiplication table to calculate

$$(\boldsymbol{i}+\boldsymbol{j})\times(\boldsymbol{j}+2\boldsymbol{k}),\ (2\boldsymbol{j}+\boldsymbol{k})\times(3\boldsymbol{i}-\boldsymbol{k})$$

and the cross product of each of these vectors with a position vector given by coordinates $(1, 2, 3)$. The position vector is to be taken as the second factor.

(Solution on p. 47.)

### SAQ A2.3

short

Which formula gives the area of a triangle bounded by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$?

(i) $|\boldsymbol{a}\times\boldsymbol{b}|$, (ii) $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$, (iii) $\boldsymbol{a}\cdot\boldsymbol{b}$.

(Solution on p. 47.)

## SAQ answers and comments

**SAQ 1**

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i}+\frac{\partial\phi}{\partial y}\boldsymbol{j}+\frac{\partial\phi}{\partial z}\boldsymbol{k}$$

$$= 4\boldsymbol{i}+3\boldsymbol{j}$$

**SAQ 2**

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i}+\frac{\partial\phi}{\partial y}\boldsymbol{j}+\frac{\partial\phi}{\partial z}\boldsymbol{k}$$

$$= 6\times 2z\boldsymbol{k}$$

$$= 12z\boldsymbol{k}$$

At $z=5$

$$\nabla\phi = 12\times 5\boldsymbol{k}$$

$$= 60\boldsymbol{k}$$

**SAQ 3**

$$\boldsymbol{F} = -\nabla V = -(2\boldsymbol{i}+8y\boldsymbol{j}-3z^2\boldsymbol{k})$$

At (2, 1, 3)

$$\boldsymbol{F} = -2\boldsymbol{i}-8\boldsymbol{j}+27\boldsymbol{k}$$

**SAQ 4** (ii)

$$\boldsymbol{F}_1 = -\nabla V_1 = -(8\boldsymbol{i}+2y\boldsymbol{j}+8\boldsymbol{k})$$

$$\boldsymbol{F}_2 = -\nabla V_2 = -(4x\boldsymbol{i}+8\boldsymbol{j}+2z\boldsymbol{k})$$

At (2, 3, 3)

$$\boldsymbol{F}_1 = -(8\boldsymbol{i}+6\boldsymbol{j}+8\boldsymbol{k})$$

$$\boldsymbol{F}_2 = -(8\boldsymbol{i}+8\boldsymbol{j}+6\boldsymbol{k})$$

Thus the directions of these vectors are different but their magnitudes are equal.

**SAQ 5** The combined potential is the sum of $V_1$ and $V_2$:

$$V_1+V_2 = 7x+4y^2+z+x^3-y+6z^2$$

At the point (3, 4, 5)

$$V_1+V_2 = 21+64+5+27-4+150$$

$$= 263$$

**SAQ 6** To find the force function, the potential functions must be differentiated according to equation G (3–2) on p. G 30.

$$F_1(x) = -\frac{\mathrm{d}V_1(x)}{\mathrm{d}x}$$

and

$$F_2(x) = -\frac{\mathrm{d}V_1(x)}{\mathrm{d}x} - \frac{\mathrm{d}C}{\mathrm{d}x}$$

$$= -\frac{\mathrm{d}V_1(x)}{\mathrm{d}x}$$

Thus

$$F_1(x) = F_2(x)$$

**SAQ 7** For the first part of the question $F(x) = -kx$.

(This is the type of force one has in simple harmonic oscillator.)

But from equation G (3–2) on p. G 30

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

Therefore

$$\frac{\mathrm{d}V(x)}{\mathrm{d}x} = kx$$

and

$$V(x) = kx^2/2 + \text{constant}$$

In the second part

$$V(x) = k/x$$

(This is the type of variation for the electrostatic potential energy due to two positive or two negative point charges. The distance $x$ is the distance between the charges.)

$$F(x) = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

$$= -\frac{\mathrm{d}(k/x)}{\mathrm{d}x}$$

$$F(x) = k/x^2$$

This is the familiar 'inverse square law' for electrostatic forces.

**SAQ 8** At a local minimum in the graph of $V(x)$ against $x$ (such as that shown in Fig. 17), the curve is horizontal i.e. $\mathrm{d}V(x)/\mathrm{d}x = 0$

Therefore from equation G (3–2) on p. G 30

$$F(x) = -\mathrm{d}V(x)/\mathrm{d}x$$
$$= 0$$

Thus the particle feels no force at $x_0$. For $x > x_0$ the slope of the graph is positive, therefore $\mathrm{d}V(x)/\mathrm{d}x$ is positive and $F(x)$ is negative (from equation G (3–2)). Thus to the right of $x_0$ the force is directed to the left. Likewise for $x < x_0$ the slope is negative, thus $\mathrm{d}V(x)/\mathrm{d}x$ is negative and $F(x)$ is positive, So to the left of $x_0$ the force is directed to the right. Thus we see that a small displacement to either side of $x_0$ results in a force tending to bring the particle back to $x_0$. This is stable equilibrium.

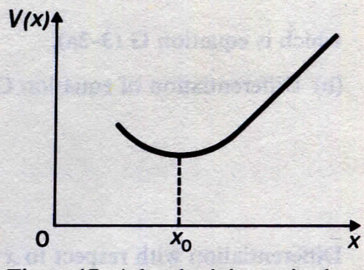



Figure 17 A local minimum in the graph of potential energy against position.

The opposite considerations apply for a local maximum: a small displacement from the equilibrium position $x_0$ results in a force directed *away* from $x_0$ and hence tends to take the particle still further away from $x_0$. In this case, $x_0$ is said to be a point of unstable equilibrium.

**SAQ 9** False.

If the state of the system is known at any chosen time, one can use the time evolution equation to discover what the state of the system was at any other time, before or after the chosen time. In any case there is no really privileged 'initial state'—it is only the state at the time one begins to observe the system.

**SAQ 10** (vi).

It is necessary to specify the velocities at $t = 0$ as well as the value of $x$ in order to answer the question.

**SAQ 11**

$$E = \tfrac{1}{2}mv^2 + V(x)$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \tfrac{1}{2}m2v\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\mathrm{d}V(x)}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

where we have been able to use the chain rule for derivatives because $V$ depends only on $x$ and $t$. Substituting for $\mathrm{d}V(x)/\mathrm{d}x$ from the definition of the potential function (equation

G (3–2)), we have

$$\frac{dE}{dt} = mv\frac{dv}{dt} - Fv$$

$$= v\left(m\frac{d^2x}{dt^2} - F\right)$$

But from Newton's second law (equation G (3–3a)), the right-hand side is zero. Therefore $dE/dt = 0$ and so $E$ is a constant.

**SAQ 12** (a) Multiplying both sides of equation G (3–7a) by $m$, we get equation G (3–3b).

Differentiation of equation G (3–7a) with respect to $t$ yields

$$\frac{d^2x}{dt^2} = \frac{1}{m}\frac{dp}{dt}$$

But from equation G (3–7b)

$$\frac{dp}{dt} = -\frac{dV(x)}{dx}$$

$$\frac{d^2x}{dt^2} = -\frac{1}{m}\frac{dV(x)}{dx}$$

and

$$-\frac{dV(x)}{dx} = F(x)$$

Thus

$$\frac{d^2x}{dt^2} = \frac{F(x)}{m}$$

which is equation G (3–3a).

(b) Differentiation of equation G (3–8) with respect to $p$ gives

$$\frac{\partial H}{\partial p} = \frac{2p}{2m} + 0$$

$$= p/m$$

Differentiation with respect to $x$ yields

$$\frac{\partial H}{\partial x} = 0 + \frac{\partial V}{\partial x}$$

$$= \frac{dV(x)}{dx}$$

**SAQ 13** Substituting the expression for $H$ from equation G (3–8) into equation G (3–10a), we have

$$\frac{dx}{dt} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + V(x)\right)$$

$$= \frac{2p}{2m} + 0$$

$$= \frac{p}{m}$$

$$p = m\,dx/dt$$

which is equation G (3–3b).

Likewise, we now substitute for $H$ in equation G (3–10b)

$$\frac{dp}{dt} = -\frac{\partial}{\partial x}\left(\frac{p^2}{2m} + V(x)\right)$$

$$= 0 - \frac{\partial V(x)}{\partial x}$$

Using the definition of $V(x)$ from equation G (3–2), we find

$$\frac{dp}{dt} = F(x)$$

But from equation G (3–3b)

$$\frac{dp}{dt} = m\frac{d^2x}{dt^2}$$

so

$$m\frac{d^2x}{dt^2} = F(x)$$

which is equation G (3–3a).

**SAQ 14** Substituting for $H$ in equation G (3–10a), we have

$$\frac{dx}{dt} = \frac{\partial}{\partial p}\left(\frac{p^2}{2m}+mgx\right)$$

$$= \frac{2p}{2m}+0$$

$$= \frac{p}{m} \qquad \text{(i)}$$

Likewise equation G (3–10b) yields

$$\frac{dp}{dt} = -\frac{\partial}{\partial x}\left(\frac{p^2}{2m}+mgx\right)$$

$$= -mg$$

Integration gives

$$p = -mgt + C_1$$

But $p=0$ at $t=0$, so $C_1=0$

$$p = -mgt$$

Substitution for $p$ in (i) yields

$$\frac{dx}{dt} = -mgt/m$$

$$= -gt$$

Integration leads to

$$x = -\tfrac{1}{2}gt^2 + C_2$$

But $x=h$ at $t=0$, so $C_2=h$

$$x-h = -\tfrac{1}{2}gt^2$$

The time taken to hit the ground is obtained by putting $x=0$ into this equation:

$$h = \tfrac{1}{2}gt^2$$
$$t = (2h/g)^{1/2}$$

**SAQ 15** The force on the planet is the gravitational attraction of the sun and this is always directed through the centre of the sun. The force, therefore, cannot give rise to any torque about this point; thus the angular momentum of the planet about the centre of the sun must be a constant.

**SAQ 16** No. The system of particles, P and S has no external forces acting on it so the torque about X is zero and hence the angular momentum of the system about X must be constant (the constant may be zero). As the contribution of P to the angular momentum of the system about X is continually changing, this means that S must also contribute to the angular momentum of the system about X: it must therefore be in motion.

In point of fact the sun and planet both revolve about a common point on the line joining their centres; this point is called the centre of mass of the system. We shall have more to say about this in Unit 2 when we come to consider a 'planetary' model of the atom.

**SAQ 17** (iv) and (v) are true.

The important point to note is that there are no external forces acting on the two-particle system, so the sum of the angular momenta of A and B about any fixed point must be a constant.

With regard to statements (i) and (ii), B acquires angular momentum about b so this must change the angular momentum of A.

With regard to statements (iii) and (iv), initially B is at rest and so has no angular momentum about b′; finally it passes through the point b′ so clearly once again it has no angular momentum about this point. Thus at these two instants of time the total angular momentum of the system b′ is that of A alone, and so the values of angular momenta at these two instants must be the same. At an intermediate time, B will have angular momentum about b′, so the value of the angular momentum of A will alter so as to keep the sum of the two constant.

As far as statement (v) is concerned, you should note that initially the angular momentum of B about a′ is zero because B is at rest; finally the angular momentum of A about a′ is zero because it passes through that point. In order to keep the sum of the angular momentum the same, the initial angular momentum of A about a′ must be equal to the final value for B about a′.

**SAQ 18** See p. E 70–73.

Your answer should be based on the following points:

(i) No
(ii) No
(iii) Equation E (3–1) on p. E 72
(iv) By using the magnetic field to cancel the effect of the electric field. See equation E (3–2) on p. E 73.
(v) He varied the composition of the cathode and the nature of the gas.

**SAQ 19** The transverse velocity $V$ on emerging from the plates is given by

$$V = at$$

where $a$ is the acceleration and $t$ is the time spent traversing the plates.

The time taken to traverse the distance $(L-l/2)$ from the end of the plates to the screen is $(L-l/2)/v$. The transverse distance travelled in this time is

$$V \times (L-l/2)/v = at(L-l/2)/v$$

The deflection $S_1S_2$ (in Figure E (3–2) on p. E 72) is given by

$$S_1S_2 = \frac{at(L-l/2)}{v} + \delta$$

Remembering that $\delta = \frac{1}{2}at^2$ and $t = l/v$ we have

$$S_1S_2 = \frac{al}{v}\left(\frac{L-l/2}{v} + \frac{l}{2v}\right) = \frac{alL}{v^2}$$

The magnification factor is
$$\frac{S_1S_2}{\delta} = \frac{alL}{v^2 \times \frac{1}{2}at^2}$$
$$= 2L/l$$

The reason Eisberg says that the value $2L/l$ is *almost* exact, is that the derivation assumes the accelerating field to be constant up to the edge of the plates and then to drop (discontinuously) to zero; this is clearly an idealization.

**SAQ 20** (i) No. Air must be present in order to provide the frictional drag on the falling droplet.
(ii) It is only by repeating the experiment many times that one discovers that the charge on the droplet $q$ never drops below a minimum value $e$ and that all other values of $q$ are an integral multiple of $e$.
(iii) The quantities that need to be measured are the distance through which the droplet falls in a given time (to obtain the terminal velocity), the voltage difference across the plates, the separation of the plates, and the density of the liquid of the droplets. There are three equations and these are used to eliminate the mass and radius of the droplet and evaluate the charge.

**SAQ 21** See p. E 90–92.

Your answer should be based on the following points:

(i) Two diaphragms.
(ii) The alpha particles are stopped in a few centimetres of air at normal atmospheric pressure.
(iii) Zinc sulphide screen.

**SAQ 22** (i) Yes. Note that unlike the situation described in SAQ 17, and shown in Figure 12, Rutherford assumes that the nucleus (particle B in that Figure) is sufficiently massive to remain fixed.
(ii) No. As the alpha particle approaches the nucleus, its kinetic energy decreases as its potential energy increases: it is its *total* energy that remains constant.
(iii) No. The foil is considered to be sufficiently thin for the energy loss due to ionization to be neglected.

**SAQ 23** Substitute $\left(\frac{\pi}{2}-\frac{\phi}{2}\right)$ for $\theta$ in equation E (4–15) on p. E 103, and replace $\sin\left(\frac{\pi}{2}-\frac{\phi}{2}\right)$ by $\cos(\phi/2)$ and $\cos\left(\frac{\pi}{2}-\frac{\phi}{2}\right)$ by $\sin(\phi/2)$:

$$\frac{1}{R}=\frac{1}{b}\cos(\phi/2)+\frac{D}{2b^2}(\sin(\phi/2)-1)$$

$$\frac{1}{R}=\frac{2}{D}\left\{\frac{D}{2b}\cos(\phi/2)+\frac{D^2}{4b^2}(\sin(\phi/2)-1)\right\}$$

From equation E (4–16) we substitute for $D/2b$:

$$\frac{1}{R}=\frac{2}{D}\left\{\tan(\phi/2)\cos(\phi/2)+\tan^2(\phi/2)(\sin(\phi/2)-1)\right\}$$

$$=\frac{2}{D}\left\{\sin(\phi/2)+\frac{\sin^2(\phi/2)(\sin(\phi/2)-1)}{(1-\sin^2(\phi/2))}\right\}$$

$$=\frac{2}{D}\left\{\sin(\phi/2)-\frac{\sin^2(\phi/2)}{1+\sin(\phi/2)}\right\}$$

$$=\frac{2}{D}\left\{\frac{\sin(\phi/2)}{1+\sin(\phi/2)}\right\}$$

Thus

$$R=\frac{D}{2}\left\{1+\frac{1}{\sin(\phi/2)}\right\}$$

**SAQ 24** See pp. E 101–103.

Your answer should be based on the following points:

(i) On the left we have the Coulomb force, and on the right the usual Newtonian acceleration, followed by the centripetal acceleration.
(ii) Equation E (4–14) on p. E 103.
(iii) $\theta \rightarrow 0$ and $dr/dt \rightarrow -v$, as $r \rightarrow \infty$.
(iv) Equation E (4–15) on p. E 103, with $A=\frac{1}{b}$ and $B=\frac{D}{2b^2}$.

**SAQ 26** We draw your attention to the assumption made in the derivation, that the rings of radius $b$ about the scattering centres did not overlap with each other. This assumption effectively puts an upper limit on the values of $b$ we are considering, and this in turn implies a minimum value of $\phi$. Thus we do not expect the formula to hold for scattering angles below this value.

Even in the absence of the overlap problem (i.e. in the idealized case of scattering from a single nucleus), one must be careful how one regards the formula. As $\phi \rightarrow 0$, so $b \rightarrow \infty$, and this implies that the radius of the scattering foil on which the alpha particles impinge must also go to infinity. If this foil of infinite radius has only a single nucleus in it, then the density $\rho$ will go to zero in the limit. Thus, once again, the probability remains bounded as $\phi \rightarrow 0$.

## Answers to SAQs set in Appendix 1

### SAQ A1.1

(i) $$f_2:(V,h) \longmapsto \sqrt{\frac{V}{\pi h}} \qquad ((V,h)\in R_0^+ \times R^+)$$

in which case $r = f_2(V,h)$

(ii) $$f_3:(V,r) \longmapsto \frac{V}{\pi r^2} \qquad ((V,r)\in R_0^+ \times R^+)$$

in which case $h = f_3(V,r)$

(iii) $$f_4:(r,h) \longmapsto 2\pi rh + 2\pi r^2 \qquad ((r,h)\in R_0^+ \times R_0^+)$$

in which case $A = f_4(r,h)$

(iv) $$f_5:(A,r) \longmapsto \frac{A}{2\pi r} - r \qquad ((A,r)\in R_0^+ \times R^+)$$

in which case $h = f_5(A,r)$

There is a sixth function, $f_6(A,h)$, which arises as the solution to a quadratic equation.

The function notation, although precise, is clearly rather cumbersome for situations of this kind, but could be very helpful in emphasizing the variables.

### SAQ A1.2

(i) $$f:x \longmapsto 1+\sin(x^2+2) \qquad (x\in R)$$

but the domain could be $[0, 1]$; we normally just assume the 'largest' domain as in (ii)

(ii) $$g:x \longmapsto \frac{x^2+1}{x-1} \qquad (x\in R, x\neq 1)$$

(iii) We have

$$f:x \longmapsto v \qquad (x\in R)$$

$$g:t \longmapsto x \qquad (t\in R_0^+)$$

therefore

$$(f\circ g):t \longmapsto v \qquad (t\in R_0^+)$$

and the function which maps time to velocity is the composite function $f\circ g$.

**SAQ A1.3** B, C.

**SAQ A1.4** Either B or C; we cannot be sure which is intended. Normally we would already either have specified that $X$ is a function in the strict mathematical sense or else indicated that it was a variable.

## Answers to SAQs set in Appendix 2

**SAQ A2.1** Your multiplication table should look like this:

| | $i$ | $j$ | $k$ |
|---|---|---|---|
| $i$ | $0$ | $k$ | $-j$ |
| $j$ | $-k$ | $0$ | $i$ |
| $k$ | $j$ | $-i$ | $0$ |

**SAQ A2.2**

$$\begin{aligned}(\boldsymbol{i}+\boldsymbol{j})\times(\boldsymbol{j}+2\boldsymbol{k}) &= \boldsymbol{i}\times\boldsymbol{j}+2\boldsymbol{i}\times\boldsymbol{k}+\boldsymbol{j}\times\boldsymbol{j}+2\boldsymbol{j}\times\boldsymbol{k}\\ &= \boldsymbol{k}-2\boldsymbol{j}+\boldsymbol{0}+2\boldsymbol{i}\\ &= 2\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}\end{aligned}$$

Likewise

$$\begin{aligned}(2\boldsymbol{j}+\boldsymbol{k})\times(3\boldsymbol{i}-\boldsymbol{k}) &= 6\boldsymbol{j}\times\boldsymbol{i}-2\boldsymbol{j}\times\boldsymbol{k}+3\boldsymbol{k}\times\boldsymbol{i}-\boldsymbol{k}\times\boldsymbol{k}\\ &= -6\boldsymbol{k}-2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{0}\\ &= -2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k}\end{aligned}$$

The position vector is $\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$

The cross product of the first vector with this position vector is

$$\begin{aligned}(2\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k})\times(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}) &= 2\boldsymbol{i}\times\boldsymbol{i}+4\boldsymbol{i}\times\boldsymbol{j}+6\boldsymbol{i}\times\boldsymbol{k}-2\boldsymbol{j}\times\boldsymbol{i}-4\boldsymbol{j}\times\boldsymbol{j}\\ &\quad -6\boldsymbol{j}\times\boldsymbol{k}+\boldsymbol{k}\times\boldsymbol{i}+2\boldsymbol{k}\times\boldsymbol{j}+3\boldsymbol{k}\times\boldsymbol{k}\\ &= \boldsymbol{0}+4\boldsymbol{k}-6\boldsymbol{j}+2\boldsymbol{k}+\boldsymbol{0}-6\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{i}+\boldsymbol{0}\\ &= -8\boldsymbol{i}-5\boldsymbol{j}+6\boldsymbol{k}\end{aligned}$$

Likewise the cross-product of the second vector with the position vector is

$$\begin{aligned}(-2\boldsymbol{i}+3\boldsymbol{j}-6\boldsymbol{k})\times(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}) &= -2\boldsymbol{i}\times\boldsymbol{i}-4\boldsymbol{i}\times\boldsymbol{j}-6\boldsymbol{i}\times\boldsymbol{k}+3\boldsymbol{j}\times\boldsymbol{i}+6\boldsymbol{j}\times\boldsymbol{j}\\ &\quad +9\boldsymbol{j}\times\boldsymbol{k}-6\boldsymbol{k}\times\boldsymbol{i}-12\boldsymbol{k}\times\boldsymbol{j}-18\boldsymbol{k}\times\boldsymbol{k}\\ &= \boldsymbol{0}-4\boldsymbol{k}+6\boldsymbol{j}-3\boldsymbol{k}+\boldsymbol{0}+9\boldsymbol{i}-6\boldsymbol{j}+12\boldsymbol{i}\\ &= 21\boldsymbol{i}-7\boldsymbol{k}\end{aligned}$$

**SAQ A2.3** The area of a triangle is $\frac{1}{2}\times\text{base}\times\text{perpendicular height} = \frac{1}{2}|\boldsymbol{b}|\,|\boldsymbol{a}|\sin\theta$ where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Therefore the correct answer is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$. Note that an area is not a vector quantity, so the modulus has been taken.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## UNIT TITLES SM 351

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